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Eulerian Parametrization of Wigner's Little Groups and Gauge Transformations in term of Rotations of Two-component Spinors

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Abstract

A set of rotations and Lorentz boosts is presented for studying the three-parameter little groups of the Poincaré group. This set constitutes a Lorentz generalization of the Euler angles for the description of classical rigid bodies. The concept of Lorentz-generalized Euler rotations is then extended to the parameterization of the $E(2)$-like little group and the $O(2,1)$-like little group for massless and imaginary-mass particles respectively. It is shown that the $E(2)$-like little group for massless particles is a limiting case of the $O(3)$-like or $O(2,1)$-like little group. A detailed analysis is carried out for the two-component $SL(2,c)$ spinors. It is shown that the gauge degrees of freedom associated with the translation-like transformation of the $E(2)$-like little group can be traced to the $SL(2,c)$ spins which fail to align themselves to their respective momenta in the limit of large momentum

	Massive Slow	between	Massless Fast
Energy Momentum	$E=\frac{p^2}{2m}$	Einstein's $E=\sqrt{m^2+p^2}$	$E = p$
Spin, Gauge Helicity	S_3 S_1 S_2	Wigner's Little Group	S_3 Gauge Trans.

1 Introduction

The Euler angles consitute a convenient parameterization of the three-dimensional rotation group. The Euler kinematics consists of two rotations around the z axis with one rotation around the y axis between them. The first question we would like to address in this paper is what happens if we add a Lorentz boost along the z direction to this traditional procedure. Since the rotation around the z axis is not affected by the boost along the same axis, we are asking what is the Lorentz-generalized form of the rotation around the y axis.

Since the publication of Wigner's fundamental paper on the Poincaré group in 1939 [1], a number of mathematical techniques have been developed to deal with the three-parameter little groups which leave a given four-momentum invariant. Our second question is why we do not yet have a standard set of transformations for Wigner's little groups.

In this paper, we combine the first and second questions. One of Wigner's little groups is locally isomprphic to $O(3)$. Furthermore, the Euler angles constitute the natural language for spinning tops in classical mechanics, while Wigner's little groups are the description of the internal space-time symmetries of relativistic particles, including spins. It is thus quite natural for us to look for a possible Eulerian parameterization of the three-parameter little groups.

As far as massive particles are concerned, the traditional approach to this problem is to go to the Lorentz frame in which the particle is at rest, and then perform rotations there [1]. Then, its four-momentum is not affected, but the direction of its spin becomes changed. This operation however is not applicable to the case of massless or imaginary-mass particles.

In order to construct a Lorentz kinematics which include both massive and massless particles, we observe that the transformation which changes a given four-momentum can be carried out in many different ways. However, as Wigner observed in 1957, the resulting spin orientation depends on the way in which the transformation is performed and on the mass of the particle [2]. For instance, when a particle with positive helicity is rotated, the helicity remains unchanged. As far as the momentum is concerned, we can achieve the same purpose by performing a simple boost. However, this boost does not leave the helicity invariant. Furthermore, the change in the direction of spin depends on the mass.

Indeed, the difference between the rotation and boost was studied for massless photons by Kupersztych [3] who observed that this difference amounts to a gauge transformation. In this paper, we extend the kinematics of Kupersztych to include massive and imaginary-mass particles. We shall show then that this extended kinematics constitutes the above-mentioned Lorentz generalization of the Euler rotations.

We then study the extended Kupersztych kinematics using the $SL(2,c)$ spinors. Among the four two-component $SL(2,c)$ spinors, two of them preserve the helicity under boosts in the zero-mass limit, as was noted by Wigner in 1957. However, the remaining two do not preserve the helicity in the same limit. We show that these helicity non-preserving spinors are responsible for gauge degrees of freedom contained in the $E(2)$-like little group for photons.

In Sec. 2, we work out the Kupersztych kinematics for massive particles. It is pointed out that this new kinematics is equivalent to the traditional $O(3)$-like kinematics in which the particle is rotated in its rest frame. We show in Sec. 3 that the $E(2)$-like little group for massless particles is the infinite-momentum/zero-mass limit of the $O(3)$-like little group discussed in Sec. 2. In Sec. 4, we discuss the continutation of the transformation matrices for the $O(3)$-like little group to the case of imaginary-mass particles.

In Sec. 5, we study the transformation properties of the four two-component spinors in the $SL(2,c)$ regime. It is shown that in the limit of infinite momentum and/or zero mass, two of the $SL(2,c)$ spinors preserve their respective helicities, while the remaining two do not. We note in Sec.

6, that four-vectors can be constructed from the four two-component $SL(2,c)$ spinors. It is shown that the origin of the gauge degrees of freedom for photons can be traced to the spinors which refuse to align themselves to the momentum in the infinite-momentum/zero-mass limit.

2 Kinematics of the O(3)-like Little Group

The Euler rotation consists of a rotation around the y axis preceded and followed by rotations around the z axis. If the boost is made along the z axis, the rotations around the z axis are not affected. In this section, we discuss a Lorentz generalization of the rotation around the y axis and its relation to the $O(3)$-like little group for massive particles.

Let us start with a massive particle at rest whose four-momentum is

$$(0,0,0,m). \tag{1}$$

We use the four-vector convention: $x^{\mu} = (x, y, x, t)$. We can boost the above four-momentum along the z direction with velocity parameter α.

$$P = m\left(0, 0, \alpha/(1-\alpha^2)^{1/2}, 1/(1-\alpha^2)^{1/2}\right). \tag{2}$$

The four-by-four matrix which transforms the four-vector of Eq.(1) to that of Eq.(2) is

$$A(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/(1-\alpha^2)^{1/2} & \alpha/(1-\alpha^2)^{1/2} \\ 0 & 0 & \alpha/(1-\alpha^2)^{1/2} & 1/(1-\alpha^2)^{1/2} \end{pmatrix}. \tag{3}$$

Let us next rotate the four-vector of Eq.(2) using the rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

This rotation does not alter the helicity of the particle [2].

As is specified in Fig. 1, we can achieve the same result on the four-momentum by applying a boost matrix. However, unlike the rotation of Eq.(4), this boost is not a helicity-preserving transformation [2]. We can study the difference between these two transformations by taking the product of the rotation and the inverse of the boost. This inverse boost is illustrated in Fig. 1, and is represented by

$$S(\theta,\alpha) = \begin{pmatrix} 1 + 2\left((\sinh(\lambda/2)\cos(\theta/2)\right)^2 & 0 & -\left(\sinh\lambda/2\right)^2 & -(\sinh\lambda)\cos(\theta/2) \\ 0 & 1 & 0 & 0 \\ -\left(\sinh(\lambda/2)\right)^2\sin\theta & 0 & 1 + 2\left(\sinh(\lambda/2)\sin(\theta/2)\right)^2 & (\sinh\lambda)\sin(\theta/2) \\ -(\sinh\lambda)\cos(\theta/2) & 0 & (\sinh\lambda)\sin(\theta/2) & \cosh(\lambda) \end{pmatrix} \tag{5}$$

where

$$\lambda = 2\left(\tanh^{-1}[\alpha(\sin(\theta/2))]\right). \tag{6}$$

This matrix depends on the rotation angle θ and the velocity parameter α, and becomes identity matrix when the particle is at rest with $\alpha = 0$.

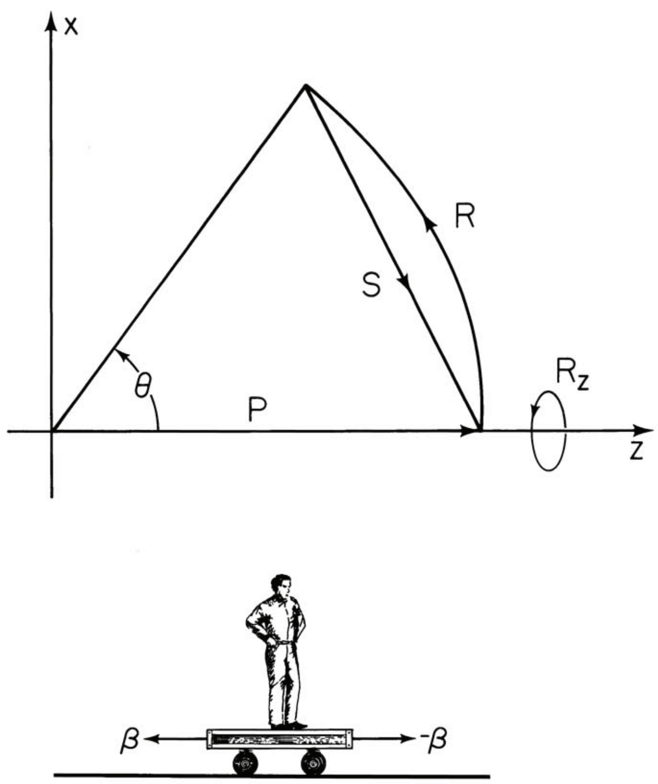


Figure 1: Lorentz-generalized Euler rotations. The traditional Euler parametrization consists of two rotations around the z axis with one rotation around the y axis between them. If we add a Lorentz boost along the z axis, the two rotations around the z axis are not affected. The rotation around the y axis can be Lorentz-generalized in the following manner. If we boost the system along the z direction, we are dealing with the system with a non-zero four-momentum along the same direction. The four-momentum p can be rotated around the y axis by angle θ. The same result can be achieved by boost S^{-1}. However, these two transformations do not produce the same effect on the spin. The most effective way of studying this difference is to study the transformation SR which leaves the initial four-momentum invariant.

Indeed, the rotation $R(\theta)$ followed by the boost $S(\alpha, \theta)$ leaves the four-momentum p of Eq.(2) invariant:

$$P = D(\alpha, \theta) P, \tag{7}$$

where $D(\alpha, \theta) = S(\alpha, \theta) R(\theta)$. The multiplication of the two matrices is straightforward, and the result is

$$D(\alpha, \theta) = \begin{pmatrix} 1-(1-\alpha^2)u^2/2T & 0 & -u/T & \alpha u/T \\ 0 & 1 & 0 & 0 \\ u/T & 0 & 0 & 0 \\ u/T & 0 & 1+u^2/2T & \alpha u^2/2T \\ \alpha u/T & 0 & -\alpha u^2/2T & 1+\alpha u^2/2T \end{pmatrix} \tag{8}$$

where $u = -2\tan(\theta/2)$, and $T = 1+(1-\alpha^2)(\tan(\theta/2))^2$. This complicated expression leaves the four-momentum P of Eq.(2) invariant. Indeed, if the particle is at rest with vanishing velocity parameter α, the above expression becomes a rotation matrix. As the velocity parameter α increases, this D matrix performs a combination of rotation and boost, but leaves the four-momentum invariant.

Let us approach this problem in the traditional framework [1]. The above transformation is clearly an element of the $O(3)$-like little group which leaves the four-momentum P invariant. Then we can boost the particle with its four-momentum P by A^{-1} until the four-momentum becomes that of Eq.(1), rotate it around the y axis, and then boost it by A until the four-momentum becomes P of Eq.(2). It is appropriate to call this rotation in the rest frame the Wigner rotation [4]. The transformation of the $O(3)$-like little group constructed in this manner should take the form:

$$D(\alpha, \theta) = A(\alpha) W(\theta^*) A(-\alpha), \tag{9}$$

where W is the Wigner rotation matrix:

$$W(\theta^*) = \begin{pmatrix} cos\theta^* & 0 & \sin\theta^* & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta^* & 0 & \cos\theta^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

We may call θ^* the Wigner angle. The question then is whether D of Eq.(9) is the same as D of Eq.(8). In order to answer this question, we first take the trace of the expression given in Eq.(9). The similarity transformation of Eq.(9) assures us that the trace of W be equal to that of D. This leads to

$$\theta^* = \cos^{-1}\left(\frac{1-(1-\alpha^2)[\tan(\theta/2)]^2}{1+(1-\alpha^2)[\tan(\theta/2)]^2}\right). \tag{11}$$

It is then a matter of matrix algebra to confirm that D of Eq.(9) and that of Eq.(8) are identical.

We have plotted in Fig. 2 the Wigner rotation angle θ^* as a function of the velocity parameter α. Here θ^* becomes θ when $\alpha = 0$, and remains approximately equal to θ when α is smaller than 0.4. Then θ^* vanishes when $\alpha \to 1$. Indeed, for a given value of θ, it is possible to determine the value of θ^* which is the rotation angle in the Lorentz frame in which the particle is at rest.

The D matrix in the traditional form of Eq.(9) is well known [1]. However, the fact that it can also be derived from the closed-loop $R(\theta)$ and $S(\alpha, \theta)$ suggests that it has a richer content. For instance, the closed-loop kinematics does not have to be unique. There is at least one other closed-loop kinematics which leaves the four-momentum invariant [5]. The Kupersztych kinematics, which we are using in this paper, is convenient for studying the relation between the Euler angles and the parameters of the $O(3)$-like little group.

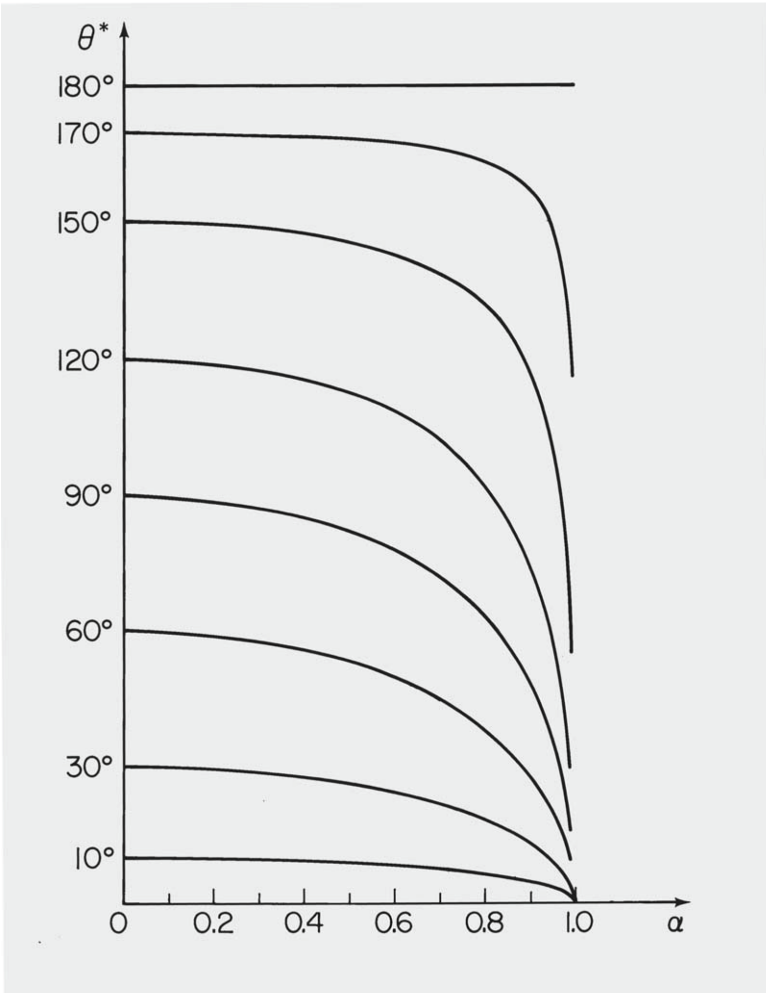


Figure 2: Wigner rotation angle versus lab-frame rotation angle. We have plotted θ^* as a fuction of α for various values of θ using Eq.(11). $\theta = \theta^*$ at $\alpha = 0.\theta^*$ is nearly equal to θ for moderate values of α, but it rapidly approaches 0 as α becomes 1.

We have so far discussed the transformations in the xz plane. It is quite clear that the same analysis can be carried out in the yz plane or any other plane containing the z axis. This means that we can perform rotations $R_z(\phi)$ and $R_z(\psi)$ respectively before and after carrying out the transformations in the xz plane. Indeed, together with the velocity parameter α, the three parameters θ, ϕ, and ψ constitute the Eulerian parametrization of the $O(3)$-like little group.

3 E(2)-like Little Group for Massless Particles

Let us study in this section the D matrix of Eq.(8) as the particle mass becomes vanishingly small, by taking the limit of $\alpha \to 1$. In this limit, the D matrix of Eq.(8) becomes

$$D(u) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1 - u^2/2 & u^2/2 \\ u & 0 & -u^2/2 & 1 + u^2/2 \end{pmatrix}. \tag{12}$$

After losing the memory of how the zero-mass limit was taken, it is impossible to transform this matrix into a rotation matrix. There is no Lorentz frame in which the particle is at rest. If we boost this expression along the z direction using the boost matrix:

$$B(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/(1-\beta^2)^{1/2} & \beta/(1-\beta^2)^{1/2} \\ 0 & 0 & \beta/(1-\beta^2)^{1/2} & 1/(1-\beta^2)^{1/2} \end{pmatrix}, \tag{13}$$

D remains form-invariant:

$$D'(u) = B(\beta)D(u)[D(\beta)]^{-1} = D(u'), \tag{14}$$

where

$$u' = [(1+\beta)/(1-\beta)]^{1/2}u$$

. The matrix of Eq.(12) is the case where the Kupersztych kinematics is performed in the xz plane. This kinematics can also be performed in the yz plane. Thus the most general form for the D matrix is

$$D(u,v) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2+v^2)/2 & (u^2+v^2)/2 \\ u & v & -(u^2+v^2)/2 & 1 + (u^2+v^2)/2 \end{pmatrix}, \tag{15}$$

The algebraic property of this expression has been discussed extensively in the literature [1, 5, 6, 7, 8]. If applied to the photon four-potential, this matrix performs a gauge transformation [6, 7]. The reduction of the above matrix into the three-by-three matrix representing a finite-dimensional representation of the two-dimensional Euclidean group has also been discussed in the literature [8].

Let us go back to Eq.(9). We have obtained the above gauge transformation by boosting the rotation matrix W given in Eq.(10). This means that the Lorentz-boosted rotation becomes a gauge transformation in the infinite-momentum and/or zero-mass limit. This observation was made earlier in terms of the group contraction of $O(3)$ to $E(2)$ [9, 10], which is a singular transformation. We are then led to the question of how the method used in this section can be analytic, while the traditional method is singular.

The answer to this question is very simple. The group contraction is a language of Lie groups [9, 10]. The parameter α we use in this paper is not a paramter of the Lie group. If we use η as the Lie-group parameter for boost along the z direction, it is related to α by $\sinh\eta = \alpha/(1-\alpha^2)^{1/2}$. However, this expression is singular at $\alpha = \pm 1$. Therefore, the continuation in α is not necessarily singular. We shall continue the discussion of this limiting process in terms of the $SL(2,c)$ spinors in Sec. 6.

4 O(2,1)-like Little Group for Imaginary-mass Particles

We are now interested in transformations which leave the four-vector of the form

$$P = im(0, 0, \alpha/(\alpha^2-1)^{1/2}, 1/(\alpha^2-1)^{1/2}) \tag{16}$$

invariant, with α greater than 1. Although particles with imaginary mass are not observed in the real world, the transformation group which leaves the above four-momentum invariant is locally isomorphic to $O(2,1)$ and plays a pivotal role in studying noncompact groups and their applications in physics. This group has been discussed extensively in the literature [11].

We are interested here in the question of whether the D matrix constructed in Secs. 2 and 3 can be analytically continued to $\alpha > 1$. Indeed, we can perform the rotation and boost of Fig. 1 to obtain the D matrix of the form given in Eq.(8), if α is smaller than α_0 where

$$\alpha_0^2 = \frac{1 + (\tan(\theta/2))^2}{(\tan(\theta/2))^2}. \tag{17}$$

As α increases, some elements of the D matrix become singular when T vanishes or $\alpha = \alpha_0$. Mathematically, this is a simple pole which can be avoided either clockwise or counterclockwise. However, the physics of this continuation process requires a more careful investigation.

One way to study the D transformation more effectivley is to boost the space-like four-vector of Eq.(16) along the z direction to a simpler vector

$$(0, 0, i, 0), \tag{18}$$

using the boost matrix of Eq.(13) with the boost parameter $\beta = 1/\alpha$. Conesequently, the D matrix is a Lorentz-boosted form of a simpler matrix F:

$$D = B(1/\alpha)F(\lambda)[B(1/\alpha)]^{-1}, \tag{19}$$

where F is a boost matrix along the x direction:

$$F(\lambda) = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cosh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}, \tag{20}$$

where

$$\begin{aligned} \tanh\lambda &= \frac{-2(\alpha^2-1)^{1/2}\tan(\theta/2)}{1 + (\alpha^2-1)[\tan(\theta/2)]^2}. \\ \cosh\lambda &= \frac{1 + (\alpha^2-1)[\tan(\theta/2)]^2}{1 - (\alpha^2-1)[\tan(\theta/2)]^2}. \end{aligned} \tag{21}$$

If we add the rotational degree of freedom around the z axis, the above result is perfectly consistent with Wigner's original observation that the little group for imaginary-mass particles is locally isomorphic to $O(2,1)$ [1].

We have observed earlier that the D matrix of Eq.(8) can be analytically continued from $\alpha = 1$ to $1 < \alpha < \alpha_o$. At $\alpha = \alpha_o$, some of its elements are singular. If $\alpha > \alpha_o, \cosh\lambda$ in Eqs. (20) and (21) become negative, and this is not acceptable.

One way to deal with this problem is to take advantage of the fact that the expression for $\tanh\lambda$ in Eq.(21) is never singular for real α greater than 1. This is possible if we change the signs of both $\sinh\lambda$ and $\cosh\lambda$ when we jump from $\alpha < \alpha_o$ to $\alpha > \alpha_o$. Indeed, the continuation is possible if it is accompanied by the reflection of x and t coordinates. After taking into accounts the reflection of the x and t coordinates, we can construct the D matrix by boosting F of Eq.(20). The expression for the D matrix for $\alpha > \alpha_0$ becomes

$$D = \begin{pmatrix} 1 - 2/T & 0 & u/T & -\alpha u/T \\ 0 & 1 & 0 & 0 \\ -u/T & 0 & 1 + 2/[(\alpha^2 - 1)T] & 2\alpha/[(\alpha^2 - 1)T] \\ -\alpha u/T & 0 & -2\alpha/[(\alpha^2 - 1)T] & 1 - 2\alpha^2/[(\alpha^2 - 1)T] \end{pmatrix}. \tag{22}$$

This expression cannot be used for the $\alpha \to 1$ limit, but can be used for the limit. In the limit $\alpha \to \infty$, P of Eq.(16) becomes identical to Eq.(18), and the above expression becomes an identity matrix. As for the question of whether D of Eq.(22) is an analytic continuation of Eq.(8), the answer is "No," because the transition from Eq.(22) to Eq.(8) requires the reflection of the x and t axes.

5 Particles with Spin 1/2

The purpose of this section is to study the D kinematics of spin-1/2 particles within the framework of $SL(2,c)$. Let us study the Lie algebra of $SL(2,c)$ [2, 3]:

$$[S_i, S_j] = i\epsilon_{ijk}S_k, \qquad [S_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}S_k. \tag{23}$$

where S_i and K_i are the generators of rotations and boosts respectively. The above commutation relations are not invariant under the sign change in S_i, but they remain invariant under the sign change in K_i. For this reason, while the generators of rotations are $S_i = \frac{1}{2}\sigma_i$, the boost generators can take two different signs: $K_i = (\pm)\frac{i}{2}\sigma_i$.

Let us start with a massive particle at rest, and the usual normalized Pauli spinors χ_+ and χ_- for the spin in the positive and negative z directions respectively. If we take into account Lorentz boosts, there are four spinors. We shall use the notation $\chi_\pm$ to which the boost generators $K_i = (+)\frac{i}{2}\sigma_i$ are applicable, and $\dot{\chi}_\pm$ to which $Ki = K_i = (-)\frac{i}{2}\sigma_i$ are applicable. There are therefore four independent $SL(2,c)$ spinors [12, 13]. In the conventional four-component Dirac equation, only two of them are independent, because Dirac equation relates the dotted spinors to the undotted counterparts. However, the recent development in supersymmetric theories [14], as well as some of more traditional approaches [15] indicates that both physics and mathematics become richer in the world where all four of $SL(2,c)$ spinors are independent. In the Appendix, we examine the nature of the restriction the Dirac equation imposes on the four $SL(2,c)$ spinors.

As Wigner did in 1957 [2], we start with a massive particles whose spin is initially along the direction of the momentum. The boost matrix which brings the $SL(2,c)$ spinors from the zero-momentum state to that of p is

$$A^{(\pm)}(\alpha) = \begin{pmatrix} [(1 \pm \alpha)/(1 \mp \alpha)]^{1/4} & 0 \\ 0 & [(1 \mp \alpha)/(1 \pm \alpha)]^{1/4} \end{pmatrix}. \tag{24}$$

where the superscripts (+) and (-) are applicable to the undotted and dotted spinors respectively. In the Lorentz frame in which the particle is at rest, there is only one rotation applicable to both sets of spinors. The rotation matrix corresponding to W of Eq.(10) is

$$W(\theta^*) = \begin{pmatrix} \cos(\theta^*/2) & -\sin(\theta^*/2) \\ \sin(\theta^*/2) & \cos(\theta^*/2) \end{pmatrix}. \tag{25}$$

where the rotation angle θ^* is given in Eq.(11).

Using the formula of Eq.(9), we can calculate the D matrix for the $SL(2,c)$ spinors. The D matrix applicable to the undotted spinors is

$$D^{(+)}(\alpha,\theta) = \begin{pmatrix} 1/\sqrt{T} & (1+\alpha)u/2\sqrt{T} \\ -(1-\alpha)u/2\sqrt{T} & 1/\sqrt{T} \end{pmatrix}. \tag{26}$$

where T and u are given in Eq.(8). The D matrix applicable to the dotted spinors is

$$D^{(-)}(\alpha,\theta) = \begin{pmatrix} 1/\sqrt{T} & (1-\alpha)u/2\sqrt{T} \\ -(1+\alpha)u/2\sqrt{T} & 1/\sqrt{T} \end{pmatrix}. \tag{27}$$

We can obtain $D^{(-)}$ from $D^{(+)}$ by changing the sign of α. Both $D^{(+)}$ and $D^{(-)}$ become W of Eq.(25) when $\alpha = 0$.

If the D transformation is applied to the $\chi_\pm$ and $\dot{\chi}_\pm$ spinors:

$$\chi'_\pm = D^{(+)}\chi_\pm, \qquad \dot{\chi}'_\pm = D^{(-)}\dot{\chi}_\pm, \tag{28}$$

the angle between the momentum and the directions of the spins represented by χ_+ and $\dot{\chi}_-$ is

$$\theta' = \tan^{-1}\{(1-\alpha)\tan(\theta/2)\}, \tag{29}$$

which becomes zero as $\alpha \to 1$. On the other hand, in the case of χ_- and $\dot{\chi}_+$, the angle becomes

$$\theta'' = \tan^{-1}\{(1+\alpha)\tan(\theta/2)\}, \tag{30}$$

In the limit of $\alpha \to 1$, this angle becomes θ_1, where

$$\theta_1 = \tan^{-1}\{2\tan(\theta/2)\}. \tag{31}$$

Indeed, the spins represented by χ_- and $\dot{\chi}_+$ refuse to align themselves with the momentum. This result is illustrated in Fig. 3.

There are D transformations for the $\alpha > 1$ case. In the special Lorentz frame in which the four-momentum takes the form of Eq.(18), the D transformation becomes that of a pure boost along the x axis:

$$F^{(\pm)}(\lambda) = \begin{pmatrix} \cosh(\lambda/2) & \pm\sinh(\lambda/2) \\ \pm\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}. \tag{32}$$

where λ is given in Eq.(21).

For $\alpha < \alpha_0$, we can continue to use $D(+)$ and $D(-)$ given in Eq.(26) and Eq.(27) respectively. However, for $\alpha > \alpha_0$, the D matrix is

$$D^{(\pm)}(\alpha,\theta) = \begin{pmatrix} (\alpha^2-1)^{1/2}[\tan(\theta/2)]/\sqrt{-T} & \pm[(\alpha\pm 1)/(\alpha\mp 1)]^{1/2}/\sqrt{-T} \\ \pm[\alpha\mp 1)/(1\pm\alpha)]^{1/2}/\sqrt{-T} & (\alpha^2-1)^{1/2}[\tan(\theta/2)]/\sqrt{-T} \end{pmatrix}. \tag{33}$$

The above expression becomes an identity matrix when $\alpha \to \infty$, as is expected from the result of Sec. 4. The D matrices of Eq.(33) are not analytic continuations of their conterparts given in Eq.(26) and Eq.(27), because the contitnuation procedure which we adopted in Sec. 4 and which we used in this section involves reflections in the x and t coordinates.

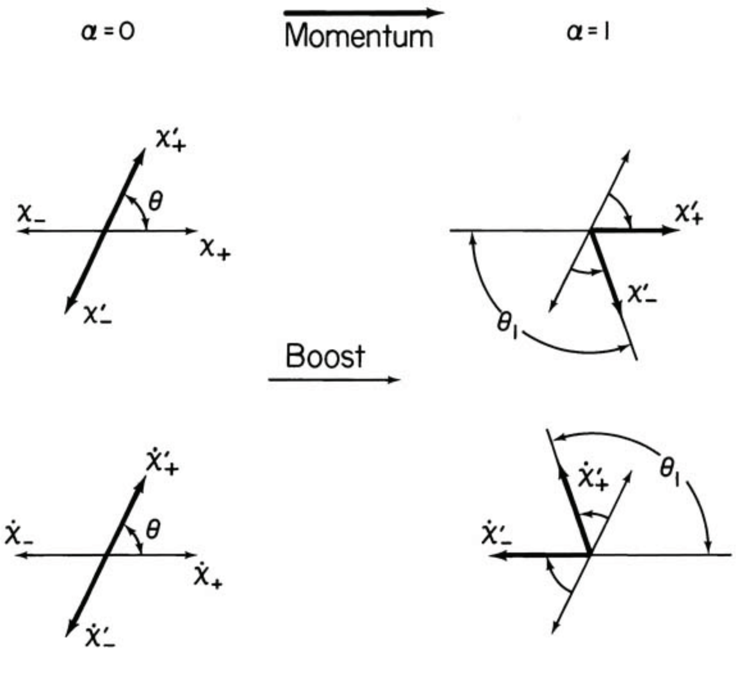


Figure 3: Lorentz-boosted rotations of the four $SL(2,c)$ spinors. If the particle velecity is zero, all the spinors rotate like the Pauli spinors. As the particle speed approaches that of light, two of the spins line up with the momentum, while the remaining two refuse to do so. Those spinors which line up are gauge invariant spinors. Those which do not are not gauge-ivariant, and they form the origin of the gauge degrees of freedom for photon four-potentials.

6 Gauge Transformations in terms of Rotations of Spinors

It is clear from the discussions of Secs. $3-5$ that the limit $\alpha \to 1$ can be defined from both directions, namely from $\alpha < 1$ and from $\alpha > 1$. In the limit $\alpha \to 1, D^{(+)}$ and $D^{(-)}$ of Eq.(26) and Eq.(27) become

$$D^{(+)} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad D^{(-)} = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}. \tag{34}$$

After going through the procedure same as that from Eq.(12) to Eq.(15), we arrive at the gauge transformation matrices [8]:

$$D^{(+)}(u,v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \qquad D^{(-)}(u,v) = \begin{pmatrix} 1 & 0 \\ -u - iv & 1 \end{pmatrix}. \tag{35}$$

applicable to the $SL(2,c)$ spinors, where $D^{(\pm)}$ are applicable to undotted and dotted spinors respectively.

The $SL(2,c)$ spinors are gauge-invariant in the sense that

$$D^{(+)}(u,v)\chi_+ = \chi_+, \qquad D^{(-)}(u,v)\chi_- = \chi_-. \tag{36}$$

On the other hand, the $SL(2,c)$ spinors are gauge-dependent in the sense that

$$D^{(+)}(u,v)\chi_- = \chi_- + (u - iv)\chi_+, \qquad D^{(-)}(u,v)\chi_+ = \chi_+ - (u + iv)\chi_-. \tag{37}$$

The gauge-invariant spinors of Eq.(36) appear as polarized neutrinos in the real world. However, where do the above gauge-dependent spinors stand in the physics of spin-1/2 particles? Are they really responsible for the gauge dependence of electromagnetic four-potentials when we construct a four-vector by taking a bilinear combination of spinors?

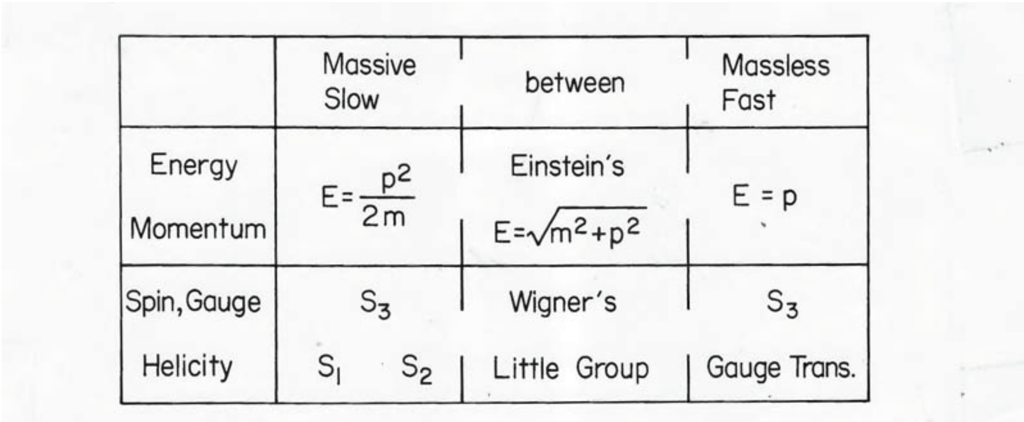
The relation between the $SL(2,c)$ spinors and the four-vectors has been discussed for massive particles. However, it is not yet known whether the same holds true for the massless case. The central issue is again the gauge transformation. The four-potentials are gauge dependent, while the spinors allowed in the Dirac equation are gauge invariant. Therefore, it is not possible to construct four-potentials from the Dirac spinors.

On the other hand, there are gauge-dependent $SL(2,c)$ spinors which are given in Eq.(37). They disappear from the Dirac spinors because N_- vanishes in the $\alpha \to 1$ limit. However, these spinors can still play an important role if they are multiplied by N_+ which neutralizes N_-. Indeed, we can construct unit vectors in the Minkowskian space by taking the direct products of two $SL(2,c)$ spinors:

$$\begin{aligned} -\chi_+\dot{\chi}_+ &= (1, i, 0, 0), \qquad \chi_-\dot{\chi}_- = (1, -i, 0, 0), \\ \chi_+\dot{\chi}_- &= (0, 0, 1, 1), \qquad \chi_-\dot{\chi}_+ = (0, 0, 1, -1). \end{aligned} \tag{38}$$

These unit vectors in one Lorentz frame are not the unit vectors in other frames. For instance, if we boost a massive particle initially at rest along the z direction, $|\chi_+\dot{\chi}_+ >$ and $\chi_-\dot{chi}_- >$ remain invariant. However, $\chi_+\dot{\chi}_- >$ and $\chi_-\dot{\chi}_+ >$ acquire the constant factors $[(1+\alpha)/(1-\alpha)]^{1/2}$ and $[(1-\alpha)/(1+\alpha)]^{1/2}$ respectively. We can therefore drop $|\chi_-\dot{\chi}_+ >$ when we go through the renormalization process of replacing the coefficient $\sqrt{(1+\alpha)/(1-\alpha)}$by 1 for particles moving with the speed of light. The $D(u,v)$ matrix for the above spinor combinations should take the form:

$$D(u,v) = D^{(+)}(u,v)D^{(-)}(u,v), \tag{39}$$



	Massive Slow	between	Massless Fast
Energy Momentum	$E = \frac{p^2}{2m}$	Einstein's $E = \sqrt{m^2 + p^2}$	$E = p$
Spin, Gauge Helicity	S_3 S_1 S_2	Wigner's Little Group	S_3 Gauge Trans.

Figure 4: Significance of the concept of Wigner's little groups. One of the beauty of Einstein's special relativity is that the energy-momentum relation for massive and slow particles and that for massless particles can be unified. Wigner's concept of the little groups unifies the internal space-time symmetries of massive and massless particles.

where $D^{(+)}$ and $D^{(-)}$ are applicable to the first and second spinors of Eq.(38) respectively. Then

$$\begin{aligned} D(u,v)|-\chi_{+}\dot{\chi}_{+}> &= |\chi_{+}\dot{\chi}_{+}> + (u+iv)|\chi_{+}\dot{\chi}_{-}>, \\ D(u,v)|-\chi_{-}\dot{\chi}_{-}> &= |\chi_{-}\dot{\chi}_{-}> + (u-iv)|\chi_{+}\dot{\chi}_{-}>, \\ D(u,v)|\chi_{+}\dot{\chi}_{-}> &= |\chi_{+}\dot{\chi}_{-}> . \end{aligned} \tag{40}$$

The first two equations of the above expression correspond to the gauge transformations on the photon polarization vectors. The third equation describes the effect of the D transformation on the four-momentum, confirming the fact that $D(u,v)$ is an element of the little group. The above operation is identical to that of the four-by-four D matrix of Eq.(15) on photon polarization vectors.

Concluding Remarks

We studied in this paper Wigner's little groups by constructing a Lorentz kinematics which leaves the four-momentum of a particle invariant. This kinematics consists of one rotation followed by one boost. Although the net transformation leaves the four-momentum invariant, the particle spin does not remain unchanged. The departure from the original spin orientation is studied in detail.

For a massive particle, this departure can be interpreted as a rotation in the Lorentz frame in which the particle is at rest. For massless particles with spin one, the net result is a gauge transformation. For a spin 1/2 particle, there are four independent spinors as the Dirac equation indicates. As the particle mass approaches zero, the spin orientations of two of the spinors remain invariant. However, the remaining two spinors do not. It is shown that this non-invariance is the cause of the gauge degree of freedom massless particles with spin 1.

In 1957, Wigner considered the possibility of unifying the internal space-time symmetries of massive and massless particles by noting the difference between rotations and boosts [2]. Wigner considered the scheme of obtaining the internal symmetry by taking the massless limit of the internal space-time symmetry group for massive particles. In the present paper, we have added the gauge degrees of freedom and spinors which refuse to align themselves to the momentum in the massless limit. The result of the present paper can be summarized in Fig. 4. While Einstein's special relativity unifies the energy-momentum relation for massive and massless particles, Wigner's little group unifies the internal space-time symmetry of massive and massless particles.

Acknowledgments

We are grateful to Professor Eugene P. Wigner for a very illuminating discussion on his 1957 paper2 on transformations which preserve helicity and those which do not. We would like to thank Dr. Avi I. Hauser for explaining to us the content of his paper on possible imaginary-mass neutrinos [14].

SL(2,c) Spinors and the Dirac Spinors

We pointed out in Sec. 5 that the four-component Dirac equation puts a restriction on the $SL(2,c)$ spinors. Let us see how this restriction manifests itself in the limit procedure of $\alpha \to 1$. In the Weyl representation of the Dirac equation, the rotation and boost generators take the form:

$$S_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \qquad K_i = \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}. \tag{41}$$

These generators accommodate both signs of the boost generators for the $SL(2,c)$ spinors. In this representation, γ_5 is diagonal, and its eigenvalue determines the sign of the boost generators.

In the Weyl representation, the D matrix should take the form:

$$D(u,v) = \begin{pmatrix} D^{(+)}(u,v) & 0 \\ 0 & D^{(-)}(u,v) \end{pmatrix}, \tag{42}$$

applicable to the Dirac spinors which, for the particle moving along the z direction with four-momentum p, are

$$U(\mathbf{p}) = \begin{pmatrix} N_+\chi_+ \\ \pm N_-\dot{\chi}_+ \end{pmatrix}, \qquad V(\mathbf{p}) = \begin{pmatrix} \pm N_-\chi_- \\ N_+\dot{\chi}_- \end{pmatrix}, \tag{43}$$

where the + and - signs in the above expression specify positive and negative energy states respectively. N_+ and N_- are the normalization constants, and

$$N_\pm = \left\{\frac{1 \pm \alpha}{1 \mp \alpha}\right\}^{1/4}. \tag{44}$$

As the momentum/mass becomes very large, N_-/N_+ becomes very small. From Eqs. (36) and (37), we can see that the large components are gauge-invariant while the small components are gauge-dependent. The gauge dependent component of the Dirac spinor disappears in the $\alpha \to 1$ limit, the Dirac equation becomes a pair of the Weyl equations. If we renormalize the Dirac spinors of Eq.(43) by dividing them by N_+, they become

$$U(\mathbf{p}) = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}, \qquad V(\mathbf{p}) = \begin{pmatrix} 0 \\ \dot{\chi}_- \end{pmatrix}, \tag{45}$$

for $\gamma_5 = \pm 1$ respectively. The gauge-dependent spinors disappear in the large-momentum/zero-mass limit. This is precisely why we do not talk about gauge transformations on neutrinos in the two-component neutrino theory.

The important point is that we can obtain the above decoupled form of spinors immediately from the most general form of spinors by imposing the gauge invariance. This means that the requirement of gauge invariance is equivalent to $\gamma_5 = 1$, as was suspected in Ref. 8.

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