

Wigner

Later in 1939, Wigner was interested in constructing subgroups of the Lorentz group whose transformations leave a given four-momentum invariant, and called these subsets “little groups.” Thus, Wigner’s little group consists of two-by-two matrices satisfying

$$\boldsymbol{P} = \boldsymbol{W} \boldsymbol{P} \boldsymbol{W}^\dagger. \quad (1)$$

This two-by-two $\boldsymbol{W}$ matrix is not an identity matrix, but tells about internal space-time symmetry of the particle with a given energy-momentum four-vector. This aspect was not known when Einstein formulated his special relativity in 1905.

If its determinant is a positive number, the $\boldsymbol{P}$ matrix can be brought to the form

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2)$$

corresponding to a massive particle at rest

If the determinant is negative, it can be brought to the form

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3)$$

corresponding to an imaginary-mass particle moving faster than light along the $\boldsymbol{z}$ direction, with its vanishing energy component.

If the determinant is zero, we may write $\boldsymbol{P}$ as

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (4)$$

corresponding to a massless particle moving along the $\boldsymbol{z}$ direction.

For all three of the above cases, the matrix of the form

$$\boldsymbol{Z}(\delta) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}$$

will satisfy the Wigner condition of Eq.(1). This matrix corresponds to rotations around the $\boldsymbol{z}$ axis, as is shown in the Appendix.

For the massive particle with the four-momentum of Eq.(2), the Naimark transformations with the rotation matrix of the form

$$\boldsymbol{R}(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

Table 1: Wigner's Little Groups. The little groups are the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. They thus define the internal space-time symmetries of particles. The four-momentum remains invariant under the rotation around it. In addition, they remain invariant under the following transformations. They are different for massive, massless, and superluminal particles.

Particle mass	Four-momentum	Transform matrices
massive	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$
Massless	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$
Imaginary mass	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$

also leaves the $\mathbf{P}$ matrix of Eq.(2) invariant. Together with the $\mathbf{Z}(\delta)$ matrix, this rotation matrix lead to the subgroup consisting of unitary subset of the $\mathbf{G}$ matrices. The unitary subset of $\mathbf{G}$ is $\mathbf{SU}(2)$ corresponding to the three-dimensional rotation group dictating the spin of the particle.