Unitary representations of the Lorentz group

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Certain quantities are introduced which are like tensors in space-time with an infinite enumerable number of components and with an invariant positive definite quadratic form for their squared length. Some of the main properties of these quantities are dealt with, and some applications to quantum mechanics are pointed out.

1. Introduction

Given any group, an important mathematical problem is to get a matrix representation of it, which means to make each element of the group correspond to a matrix in such a way that the matrix corresponding to the product of two elements is the product of the matrices corresponding to the factors. The matrices may be looked upon as linear transformations of the co-ordinates of a vector and then each element of the group corresponds to a linear transformation of a field of vectors. Of special interest are the unitary representations, in which the linear transformations leave invariant a positive definite quadratic form in the co-ordinates of a vector.

The Lorentz group is the group of linear transformations of four real variables $t_0, t_1, t_2, t_3$, such that $-t_0^2 - t_1^2 - t_2^2 - t_3^2$ is invariant. The finite representations of this group, i.e. those whose matrices have a finite number of rows and columns, are all well known, and are dealt with by the usual tensor analysis and its extension spinor analysis. None of them is unitary. The group has also some infinite representations which are unitary. These do not seem to have been studied much, in spite of their possible importance for physical applications.

The present paper gives a new method of attack on these representations, which was suggested by Fock’s quantum theory of the harmonic oscillator. It leads to a new kind of tensor quantity in space-time, with an infinite number of components and a positive definite expression for its squared length.

2. Three-dimensional theory

This section will be devoted to some preliminary work applying to the rotation group of three-dimensional Euclidean space. Take an ascending power series

$$a_0 + a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3 + \ldots$$

in a real variable $\xi_i$ with real coefficients $a_r$. Consider these coefficients to be the co-ordinates of a vector in a certain space of an infinite number of dimensions, and define the squared length of the vector to be

$$\sum_0^x r! a_r^2.$$ 

The series (2) must converge for the vector to be a finite one.
Take two more similar power series \( \sum_0^\infty b_r \xi^r_2 \) and \( \sum_0^\infty c_r \xi^r_3 \) in the real variables \( \xi_2 \) and \( \xi_3 \) and consider their coefficients to be the co-ordinates of vectors in two more vector spaces, with squared lengths defined by the corresponding formula to (2). Now multiply the three vector spaces together. A general vector in the product space will be a sum of products of vectors of the three original vector spaces, and its co-ordinates \( A_{rst} \) can be represented as the coefficients in a power series

\[
P = \sum_0^\infty A_{rst} \xi^r_1 \xi^s_2 \xi^t_3
\]

in the three variables \( \xi_1, \xi_2, \xi_3 \). The squared length of such a vector is

\[
\sum_0^\infty r! s! t! A^2_{rst},
\]

and two vectors with co-ordinates \( A_{rst} \) and \( B_{rst} \) have a scalar product

\[
\sum_0^\infty r! s! t! A_{rst} B_{rst}.
\]

If the variables \( \xi_1, \xi_2, \xi_3 \) are subjected to a linear transformation, going over into \( \xi'_1, \xi'_2, \xi'_3 \), say, the power series (3) will go over into a power series in \( \xi'_1, \xi'_2, \xi'_3 \),

\[
P = \sum A'_{rst} \xi'^r_1 \xi'^s_2 \xi'^t_3,
\]

in which the coefficients \( A' \) are linear functions of the previous coefficients \( A \). Thus each linear transformation of the \( \xi \)'s generates a linear transformation of the coefficients \( A \).

The theorem will now be proved: A linear transformation of \( \xi_1, \xi_2, \xi_3 \) which leaves \( \xi_1^2 + \xi_2^2 + \xi_3^2 \) invariant generates a linear transformation of the coefficients \( A \) which leaves the squared length (4) invariant. Consider first the infinitesimal transformation

\[
\xi' = \xi + \epsilon \xi', \quad \xi'_2 = \xi'_2 - \epsilon \xi'_1, \quad \xi'_3 = \xi'_3,
\]

which leaves \( \xi_1^2 + \xi_2^2 + \xi_3^2 \) invariant, \( \epsilon \) being a small quantity whose square is negligible. Substituting into (3), one gets

\[
P = \sum A_{rst} \xi'^r_1 \xi'^s_2 \xi'^t_3 + \epsilon \sum A_{rst} \xi'^r_1 \xi'^s_2 \xi'^t_3 - \epsilon \sum A_{rst} \xi'^r_1 \xi'^s_2 \xi'^t_3 - \epsilon \sum A_{rst} \xi'^r_1 \xi'^s_2 \xi'^t_3.
\]

Hence

\[
A'_{rst} = A_{rst} + (r + 1) \epsilon A_{r+1,s-1,t} - (s + 1) \epsilon A_{r-1,s+1,t},
\]

in which \( A_{rst} \) with a negative suffix is counted as zero. Thus

\[
\sum r! s! t! A'_{rst} = \sum r! s! t! [A_{rst} + 2(r + 1) \epsilon A_{r+1,s-1,t} - 2(s + 1) \epsilon A_{r-1,s+1,t} - A_{rst}].
\]

The last two terms in the [ ] here cancel, as may be seen by substituting \( r - 1 \) for \( r \) in the former and \( s - 1 \) for \( s \) in the latter, and hence the squared length (4) is invariant for the transformation (6). Any linear transformation of \( \xi_1, \xi_2, \xi_3 \) which leaves \( \xi_1^2 + \xi_2^2 + \xi_3^2 \) invariant can be built up from the infinitesimal transformation (6) and similar infinitesimal transformations with \( \xi_1, \xi_2 \) and \( \xi_3 \) permuted, together with possibly a reflexion \( \xi_1 = -\xi'_1, \xi_2 = \xi'_2, \xi_3 = 0 \), which obviously leaves the squared length (4) invariant, and hence the theorem is proved.

The group of transformations of the \( \xi \)'s which leave \( \xi_1^2 + \xi_2^2 + \xi_3^2 \) invariant is the rotation group in three-dimensional Euclidean space, so the transformations of the
coefficients $A$ provide a representation of this rotation group. One may restrict the function $P$ to be homogeneous, of degree $u$ say, and then the representation is a finite one. The coefficients $A$ then form the components of a symmetrical tensor of rank $u$, the connexion with the usual tensor notation being effected by taking $A_{rst}$ to be $u!/r!s!t!$ times the usual tensor component with the suffix $1$ occurring $r$ times, $2$ occurring $s$ times and $3$ occurring $t$ times, as may be seen from the invariance of expression (3) with $(\xi_1, \xi_2, \xi_3)$ transforming like a vector.

One can make a straightforward generalization of the foregoing theory by introducing other triplets of variables, $\eta_1, \eta_2, \eta_3$ and $\xi_1, \xi_2, \xi_3$ say, which transform together with $\xi_1, \xi_2, \xi_3$, and setting up a power series in all the variables. The transformations of the coefficients of these more general power series will provide further representations of the three-dimensional rotation group. If such a more general power series is restricted to be homogeneous, its coefficients will form the components of an unsymmetrical tensor.

3. Four-dimensional theory

Take a descending power series

$$k_0/\xi_0 + k_1/\xi_0^2 + k_2/\xi_0^3 + k_3/\xi_0^4 + \ldots$$

in a real variable $\xi_0$ with real coefficients $k_n$. Consider these coefficients to be the co-ordinates of a vector in a certain space of an infinite number of dimensions, and define the squared length of the vector to be

$$\Sigma_0^\infty k_n^2/n!.$$  \hspace{1cm} (8)

Multiply this vector space into the vector space of the preceding section. A general vector in the product space will have co-ordinates $A_{n rst}$ which can be represented as the coefficients in a power series

$$Q = \Sigma_0^\infty A_{n rst} \xi_0^{-n-1} r! s! t! \xi_1^{r} \xi_2^{s} \xi_3^{t}$$

in the four variables $\xi_0, \xi_1, \xi_2, \xi_3$. The squared length of such a vector is

$$\Sigma_0^\infty n!^{-1} r! s! t! A_{n rst}^2,$$  \hspace{1cm} (10)

and two vectors with co-ordinates $A_{n rst}$ and $B_{n rst}$ have a scalar product

$$\Sigma_0^\infty n!^{-1} r! s! t! A_{n rst} B_{n rst}.$$  \hspace{1cm} (11)

The series (7) may be extended backwards to include some terms with non-negative powers of $\xi_0$, so that coefficients $k_n$ occur with negative $n$-values, leading to coefficients $A_{n rst}$ with negative $n$-values. Since $n!$ is infinite for negative $n$, these new coefficients do not contribute to the squared length of a vector or the scalar product of two vectors. Thus the terms with non-negative powers of $\xi_0$ should be counted as corresponding to the vector zero, and whether they are present in an expansion or not does not matter.
Now apply a Lorentz transformation to the $\xi'$s,

$$\xi'_\mu = \alpha^\mu_\nu \xi'_\nu,$$

(12)

the $\alpha$'s satisfying certain conditions so that $\xi'_0 - \xi'_1 - \xi'_2 - \xi'_3$ is invariant. This makes $\xi'_0 \xi'_0 \xi'_0 \xi'_3$ go over into a finite polynomial in $\xi'_0$, $\xi'_1$, $\xi'_2$, $\xi'_3$, and $\xi'_0^{-1}$ go over into

$$(\alpha'_0 \xi'_0 + \alpha'_0 \xi'_1 + \alpha'_0 \xi'_2 + \alpha'_0 \xi'_3)^{-1},$$

(13)

which may be expanded in ascending powers of $\xi'_1$, $\xi'_2$, $\xi'_3$ and descending powers of $\xi'_0$. (The question of the convergence of this expansion is left to the next section, so as not to break the main argument here.) The power series (9) then goes over into a series

$$Q = \sum A'_{nrs} \xi'^{-n-1}_0 \xi'^r \xi'^s \xi'^t,$$

(14)

in ascending powers of $\xi'_1$, $\xi'_2$, $\xi'_3$ and descending powers of $\xi'_0$, with coefficients $A'$ which are linear functions of the previous coefficients $A$. There may be terms in (14) with non-negative powers of $\xi'_0$, but on account of what was said above these can be discarded. If such terms are present in the original series (9), they will not affect any of the coefficients with non-negative $n$-values in (14).

The Lorentz transformation thus generates a linear transformation of the coefficients $A'_{nrs}$ with non-negative $n$-values. The theorem will now be proved: The transformation of the coefficients $A$ leaves the squared length (10) invariant.

Consider first the infinitesimal Lorentz transformation

$$\xi'_0 = \xi'_0 + e\xi'_1, \quad \xi'_1 = \xi'_1 + e\xi'_0, \quad \xi'_2 = \xi'_2, \quad \xi'_3 = \xi'_3.$$  

(15)

Substituting into (9), one gets

$$Q = \sum A_{nrs} (\xi'^{-n-1}_0 \xi'^r \xi'^s \xi'^t - (n+1) e\xi'^{r-2}_0 \xi'^r \xi'^s \xi'^t + \frac{e^2 \xi'^{r-1}}{\xi'^s \xi'^t}).$$

Hence

$$A'_{nrs} = A_{nrs} - n e A_{n-1,r-1,s,t} r e A_{n+1,r+1,s,t}.$$  

in which $A_{nrs}$ with a negative value for $r$, $s$ or $t$ is counted as zero. Thus

$$\sum n! - 1 r! s! t! A'^2_{nrs}$$

$$= \sum n! - 1 r! s! t! \frac{A^2_{nrs} - 2 n e A_{nrs} A_{n-1,r-1,s,t} + 2 (r+1) e A_{n+1,r+1,s,t}}{A_{nrs}}.$$  

The last two terms in the [] here cancel, as may be seen by substituting $n+1$ for $n$ in the former and $r-1$ for $r$ in the latter, and hence the squared length (10) is invariant for the transformation (15). Any Lorentz transformation can be built up from infinitesimal transformations like (15) and three-dimensional rotations like those considered in the preceding section, together with possibly a reflexion, which obviously leaves the squared length (10) invariant, and hence the theorem is proved.

The transformations of the coefficients $A$ thus provide a unitary representation of the Lorentz group. The coefficients themselves form the components of a new kind of tensor quantity in space-time. I propose for it the name expansor, because of its connexion with binomial expansions. One may restrict the function (9) to be homogeneous and one then gets a simpler kind of expansor, which may be called a
homogeneous expander. The analogy with the three-dimensional case suggests that one should look upon a homogeneous expander as a symmetrical tensor in space-time with the suffix 0 occurring in its components a negative number of times.

The foregoing theory can be generalized, like the three-dimensional theory, by the introduction of other quadruplets of variables, \(\eta_0, \eta_1, \eta_2, \eta_3\) and \(\xi_0, \xi_1, \xi_2, \xi_3\) say, which transform together with \(\xi_0, \xi_1, \xi_2, \xi_3\). One can then set up a power series in all the variables, ascending in those variables with suffixes 1, 2 and 3, and descending in those variables with suffix 0. The transformations of the coefficients of these more general power series will provide further unitary representations of the Lorentz group, and the coefficients themselves will form the components of more general expanders.

There is another generalization which may readily be made in the theory, namely, to take the values of the index \(n\) in (9) to be not integers, but any set of real numbers \(n_0, n_0 + 1, n_0 + 2, \ldots\) extending to infinity. In the formula (10) for the squared length \(n\) is then to be interpreted as \(\Gamma(n + 1)\). The terms with negative \(n\)-values can no longer be discarded. The expression for the squared length is still positive definite if the minimum value of \(n\) is greater than \(-1\), which is the case if the function (9) is homogeneous and its degree is negative. The resulting representation is then still unitary. If, however, the function (9) is homogeneous and its degree is positive, there will be a finite number of negative terms in the expression for the squared length. The resulting kind of representation may be called nearly unitary.

4. SOME THEOREMS ON CONVERGENCE

If the series (2) is convergent, then (1) is convergent for all values of \(\xi_1\). Similarly, if (4) is convergent, then (3) is convergent for all values of \(\xi_1, \xi_2, \xi_3\). On the other hand, if (8) is convergent, (7) need not be convergent for any value of \(\xi_0\), in which case, of course, it does not define a function of \(\xi_0\). Similarly, if (10) is convergent, (9) need not be convergent for any values for the \(\xi\)'s. Thus, corresponding to a general expander \(A_{nrd}\), there need not exist any function \(Q\) of the \(\xi\)'s. However, it will now be proved that if the series (9) is homogeneous and (10) is convergent, then (9) is absolutely convergent for all values of the \(\xi\)'s satisfying

\[
\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 > 0.
\]  

(16)

If the series (9) is homogeneous of degree \(u - 1\), it may be written

\[
\sum_{n=0}^{\infty} \frac{\xi^n}{n!} = \sum_{r,s,t} A_{nrd} \frac{\xi^r \xi^s \xi^t}{r!s!t!}.
\]  

(17)

where \(\Sigma_S\) means a sum over all values of \(r, s\) and \(t\) satisfying

\[
r + s + t = n + u.
\]  

(18)

With this notation the series (10) may be written

\[
\sum_{n=0}^{\infty} \frac{\xi^n}{n!} = \sum_{r,s,t} A^2_{nrd}.
\]  

(19)
Now apply Cauchy's inequality
\[(x_1y_1 + x_2y_2 + x_3y_3 + \ldots)^2 \leq (x_1^2 + x_2^2 + x_3^2 + \ldots)(y_1^2 + y_2^2 + y_3^2 + \ldots)\] (20)
in the following way. Take \((r + s + t)!r!s!t!\) of the \(x\)'s in (20) to be each equal to \(\xi_1\xi_2\xi_3\) and the corresponding \(y\)'s to be each equal to \(A_{rst}\), and do this for all \(r, s, t\) subject to (18) for a fixed value of \(n\). Then (20) becomes
\[(\sum_{S} A_{rst} \xi_1 \xi_2 \xi_3)^2 \leq (\xi_1^2 + \xi_2^2 + \xi_3^2)^{n+u}(n+u)!^{-1} \sum_{S} r!s!t! A_{rst}^2.\] (21)
If the sum with respect to \(n\) in (19) converges, there must be some number \(\kappa\) such that
\[n!^{-1} \sum_{S} r!s!t! A_{rst}^2 < \kappa\]
for all \(n\). Then from (21)
\[\left| \xi_0^{-n-1} \sum_{S} A_{rst} \xi_1 \xi_2 \xi_3 \right| < \kappa \left| \xi_0 \right|^{u-1} \left( \frac{n!}{(n+u)!} \right)^{\frac{1}{2}} \left( \xi_1^2 + \xi_2^2 + \xi_3^2 \right)^{\frac{1}{2}(n+u)} \]
which shows that (17) is convergent when (16) is satisfied. One may take \(\xi_0, \xi_1, \xi_2, \xi_3\) and the \(A\)'s to be all positive without disturbing the argument, so (17), considered as a quadruple series in \(n, r, s, t\), is absolutely convergent.

Thus for any homogeneous expansor of finite length, there exists a function \(Q\) given by (9) defined within the light-cone (16). In transforming this function with a Lorentz transformation (12), it is legitimate to use the expansion of (13) in ascending powers of \(\xi_1, \xi_2, \xi_3\) and descending powers of \(\xi_0\), for the following reason. Suppose \(\xi_0\) is within the light-cone and take for definiteness \(\xi_0 > 0\). Then
\[a_0^0 \xi_0^0 + a_0^1 \xi_0^1 + a_0^2 \xi_0^2 + a_0^3 \xi_0^3 > 0.\] (22)
One can change the sign of any of the co-ordinates \(\xi_1, \xi_2, \xi_3\) leaving \(\xi_0\) unchanged, and \(\xi_0\) will still lie within the light-cone with \(\xi_0 > 0\), so (22) must still be satisfied.

Hence
\[a_0^0 \xi_0^0 - |a_0^1 \xi_0^1| - |a_0^2 \xi_0^2| - |a_0^3 \xi_0^3| > 0,\]
which shows that the expansion of (13) in the required manner is absolutely convergent. The use of this expansion in the preceding section is thus justified for the case when (9) is homogeneous, the new coefficients \(A\) being determined by the transformed function \(Q\) within the light-cone. The justification for (9) not homogeneous then follows since terms of different degree do not interfere. It should be noted that all the foregoing arguments are valid also when the values of \(n\) are not integers.

There are some expansors which are invariant under all Lorentz transformations, namely those whose components are the coefficients of \((\xi_0 - \xi_1 - \xi_2 - \xi_3)^l\) expanded in ascending powers of \(\xi_1, \xi_2, \xi_3\) and descending powers of \(\xi_0\). For such an expansion to be possible, \(l\) must not be a negative integer or zero, but it can be any other real number. Again, there are expansors which transform like ordinary tensors under Lorentz transformations, namely those whose components are the coefficients in the expansion of
\[f(\xi_0, \xi_1, \xi_2, \xi_3) (\xi_0 - \xi_1 - \xi_2 - \xi_3)^l.\] (23)
where \( f \) is any homogeneous integral polynomial in \( \xi_0, \xi_1, \xi_2, \xi_3 \), and \( l \) is restricted as before. (23) transforms like the polynomial \( f \), and thus like a tensor of order equal to the degree of \( f \). One would expect all these expansors to be of infinite length, as otherwise one could set up a positive definite form for the squared length of a tensor in space-time. A formal proof that they are is as follows.

Suppose the \( f \) in (23) is of degree \( u \) and express it as

\[
f = g_u + \xi_0 g_{u-1} + (\xi_0^2 - \xi_2) g_{u-2} + \xi_0 (\xi_0^2 - \xi_2^2) g_{u-3} + (\xi_0^2 - \xi_2^2)^2 g_{u-4} + \ldots,
\]

where \( \xi_2^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \) and each of the \( g \)'s is a polynomial in \( \xi_1, \xi_2, \xi_3 \) only, of degree indicated by the suffix. The successive terms here contribute to (23) the amounts

\[
g_u(\xi_0^2 - \xi_2^2)^{-l} = \sum_{m=0}^{\infty} \frac{l(l+1)(l+2)\ldots(l+m-1)g_u \xi_2^{2m}}{\xi_0^{2(m+l)}},
\]

\[
g_{u-1}\xi_0(\xi_0^2 - \xi_2^2)^{-l} = \sum_{m=0}^{\infty} \frac{l(l+1)(l+2)\ldots(l+m-1)g_{u-1} \xi_2^{2m}}{\xi_0^{2(m+l)-1}},
\]

\[
g_{u-2}(\xi_0^2 - \xi_2^2)^{-l+1} = \sum_{m=-1}^{\infty} \frac{(l-1)(l+1)\ldots(l+m-1)g_{u-2} \xi_2^{2(m+1)}}{\xi_0^{2(m+l)}},
\]

\[
g_{u-3}(\xi_0^2 - \xi_2^2)^{-l+1} = \sum_{m=-1}^{\infty} \frac{(l-1)(l+1)\ldots(l+m-1)g_{u-3} \xi_2^{2(m+1)}}{\xi_0^{2(m+l)-1}},
\]

and so on. These expansions show that, for large values of \( m \), the terms arising from \( g_{u-2}, g_{u-4}, \ldots \) are of smaller order than the corresponding terms arising from \( g_u \), and the terms arising from \( g_{u-3}, g_{u-5}, \ldots \) are of smaller order than the corresponding terms arising from \( g_{u-1} \), so that in testing for the convergence of the series which gives the squared length, only the \( g_u \) and \( g_{u-1} \) terms need be taken into account. (It will be found that the convergence conditions are not sufficiently critical to be affected by this neglect.) Now express \( g_u \) and \( g_{u-1} \) as

\[
g_u = S_u + \xi_2 S_{u-2} + \xi_4 S_{u-4} + \ldots,
\]

\[
g_{u-1} = S_{u-1} + \xi_2 S_{u-3} + \xi_4 S_{u-5} + \ldots,
\]

where the \( S \)'s are solid harmonic functions of \( \xi_1, \xi_2, \xi_3 \), of degrees indicated by the suffixes. Each of them gives a contribution to (23) of the form

\[
S_{u-2r} \xi_0^{2r} (\xi_0^2 - \xi_2^2)^{-l} = S_{u-2r} \xi_0^{2r} \sum_{m=0}^{\infty} \frac{(m+l-1)! \xi_2^{2(m+r)}}{(l-1)! \xi_0^{2(m+l)}}
\]

or

\[
S_{u-2r-1} \xi_0 \xi_2^{2r} (\xi_0^2 - \xi_2^2)^{-l} = S_{u-2r-1} \xi_0 \xi_2^{2r} \sum_{m=0}^{\infty} \frac{(m+l-1)! \xi_2^{2(m+r)}}{(l-1)! \xi_0^{2(m+l)-1}}.
\]

Using the result (41) of the appendix, one finds for the squared lengths of the expansors whose components are the coefficients of (24) and (25), series of the form

\[
c \sum_{m} \frac{(m+l-1)! 4^{m+r}(m+r)! (m+u-r+\frac{1}{2})!}{m!^2 (l-1)!^2 (2m+2l-1)!},
\]

\[
c \sum_{m} \frac{(m+l-1)! 4^{m+r}(m+r)! (m+u-r-\frac{1}{2})!}{m!^2 (l-1)!^2 (2m+2l-2)!},
\]
respectively. In both these series the ratio of the \((m + 1)\)th term to the \(m\)th is \(1 + u/m\) for large \(m\), and since \(u\) is necessarily positive or zero, the series diverge and the expansors are of infinite length. From the result (40) of the appendix, expansors associated with solid harmonics of different degrees are orthogonal to one another, and hence the total expansor with the coefficients of (23) as components must also be of infinite length.

5. A TRANSFORMATION OF VARIABLES

Up to the present, expansors have always been considered in connexion with certain \(\xi\)-functions, the components of the expansor being the coefficients of the \(\xi\)-function. In the case of integral \(n\)-values, one can make a transformation of variables of the kind which is familiar in quantum mechanics, and get the expansors connected with some other functions, which serves to clarify some of their properties.

Introduce the four operators \(x_{\mu}\), defined by

\[
2^\perp x_0 = \xi_0 - \partial/\partial \xi_0, \quad 2^\perp x_r = \xi_r + \partial/\partial \xi_r \quad (r = 1, 2, 3).
\]

(26)

Under Lorentz transformations they transform like the components of a 4-vector. The operators \(\partial/\partial x_{\mu}\) may be taken to be

\[
2^\perp \partial/\partial x_0 = \xi_0 + \partial/\partial \xi_0, \quad 2^\perp \partial/\partial x_r = - \xi_r + \partial/\partial \xi_r,
\]

(27)

as this gives the correct commutation relations between all the operators. One can now represent \(\xi\)-functions with integral \(n\)-values by functions of the \(x\)'s according to the following scheme.

Take as starting point the \(\xi\)-function \(\xi_0^{-1}\). It vanishes when operated on by \(\partial/\partial \xi_1\), \(\partial/\partial \xi_2\), or \(\partial/\partial \xi_3\), and also effectively vanishes when multiplied by \(\xi_0\), since the result of the multiplication can be discarded. It must therefore be represented by a function of the \(x\)'s which vanishes when operated on by \(x_r + \partial/\partial x_r\) with \(r = 1, 2\) or 3, or by \(x_0 + \partial/\partial x_0\), and thus by a multiple of \(e^{-i(x_0^2 + x^2)}\), where \(x^2 = x_1^2 + x_2^2 + x_3^2\). Take

\[
\xi_0^{-1} = \pi^{-1} e^{-i(x_0^2 + x^2)},
\]

(28)

where the sign \(\equiv\) means ‘represented by’. Multiply the left-hand side of (28) by the operator \((-\partial/\partial \xi_0)^n \xi_1 \xi_2 \xi_3\) and the right-hand side by the operator equal to it according to (26) and (27). The result is, after dividing through by \(n!\),

\[
\xi_0^{-n-1} \xi_1 \xi_2 \xi_3 = \pi^{-1} n! 2^{-l(n+r)}(x_0 - \partial/\partial x_0)^n(x_1 - \partial/\partial x_1)^r(x_2 - \partial/\partial x_2)^s(x_3 - \partial/\partial x_3)^te^{-i(x_0^2 + x^2)}
\]

(29)

say. This gives the function \(F\) of the \(x\)'s which represents \(\xi_0^{-n-1} \xi_1 \xi_2 \xi_3\). A general \(\xi\)-function is now represented thus,

\[
\Sigma A_{ntst} \xi_0^{-n-1} \xi_1 \xi_2 \xi_3 = \Sigma A_{ntst} F_{ntst}(x_0, x_1, x_2, x_3).
\]

(30)

In this way a general expansor \(A_{ntst}\) gets connected with a function of the \(x\)'s.
The chief interest of this connexion is that the law for the scalar product of two
expansors becomes very simple when expressed in terms of the functions of the x's.
The scalar product is the integral of the product of the two functions of the x's over all
x₀, x₁, x₂ and x₃. To prove this, first evaluate the integral
\[ \int_{-\infty}^{\infty} \left( z - \frac{d}{dz} \right)^m e^{iz} \cdot \left( z - \frac{d}{dz} \right)^{m'} e^{-iz} dz, \]
where the dot has the meaning that operators to the left of it do not operate on
functions to the right of it. For \( m > 0 \), (31) goes over by partial integration into
\[ \int_{-\infty}^{\infty} \left( z - \frac{d}{dz} \right)^{m-1} e^{-iz} \cdot \left( z - \frac{d}{dz} \right)^{m'} e^{iz} dz \]
\[ = \int_{-\infty}^{\infty} \left( z - \frac{d}{dz} \right)^{m-1} e^{-iz} \cdot \left( z + \frac{d}{dz} \right)^{m'} e^{iz} dz + 2m' \left( z - \frac{d}{dz} \right)^{m-1} e^{iz} dz \]
\[ = 2m' \int_{-\infty}^{\infty} \left( z - \frac{d}{dz} \right)^{m-1} e^{-iz} \cdot \left( z - \frac{d}{dz} \right)^{m-1} e^{iz} dz. \]
Applying this procedure \( m \) times, one gets zero if \( m' < m \) and \( 2^m m' \pi^1 \) if \( m' = m \).
Since there is symmetry between \( m \) and \( m' \), the result is
\[ \int_{-\infty}^{\infty} \left( z - \frac{d}{dz} \right)^m e^{-iz} \cdot \left( z - \frac{d}{dz} \right)^m e^{iz} dz = 2^m m' \pi^1 \delta_{mm'}. \]
Now substitute for \( z \) each of the variables \( x₀, x₁, x₂, x₃ \) in turn, with \( m \) equal to \( n, r, s, t \)
and \( m' \) equal to \( n', r', s', t' \) respectively, and multiply the four equations so obtained.
The result, after dividing through by \( \pi^4 n!^2 2^{n+r+s+t} \), is
\[ \int \int \int \int F_{nrs} F_{n're's'} d x₀ d x₁ d x₂ d x₃ = n!^{-1} r! s! t! \delta_{nn'} \delta_{rr'} \delta_{ss'} \delta_{tt'}. \]
This proves the theorem for the case when the expansors each have only one non-
vanishing component, which is sufficient to prove it generally.

One can obviously get a unitary representation of the Lorentz group by con-
sidering the transformations of a set of vectors in a space of an infinite number of
dimensions, where each vector corresponds to a function of four Lorentz variables
\( x₀, x₁, x₂, x₃ \), and where the squared length of a vector is defined as the integral of the
square of the function over all \( x₀, x₁, x₂, x₃ \). The above work shows how this obvious
unitary representation is connected with the expansor theory.

6. Applications to Quantum Mechanics

The four \( x \)'s of the preceding section may be looked upon as the co-ordinates
of a four-dimensional harmonic oscillator, the four operators \( \partial / \partial x_μ \) being the con-
jugate momenta \( p_μ \), and the energy of the oscillator may be taken to be the Lorentz
invariant
\[ \frac{1}{2} [ x₁² + x₂² + x₃² - x₀² + (p¹)² + (p²)² + (p₃)² - (p⁰)²]. \]
(32)
The 1, 2, 3 components of the oscillator thus have positive energies and the 0 com-
ponent negative energy.
The state of the oscillator for which the 0, 1, 2, 3 components are in the \( n \), \( r \), \( s \), \( t \)th quantum states respectively is then represented by the function \( F_{nrd} \) defined by (29), with a suitable normalizing factor. This representative may be transformed to the \( \xi \)-representation and becomes \( \xi_n^{-1} \xi_r \xi_s \xi_t \). Thus a state of the oscillator for which each of its components is in a quantum state corresponds to an expensor with one non-vanishing component. A general state of the oscillator therefore corresponds to a general expensor with integral \( n \)-values. A stationary state of the oscillator corresponds to a homogeneous expensor, the degree of the expensor giving the energy of the state with neglect of zero-point energy.

Four-dimensional harmonic oscillators of the above type occur in the theory of the electromagnetic field. Each Fourier component of the field, specified by a particular frequency and a particular direction of motion of the waves, provides one such oscillator, its four components coming from the four electromagnetic potentials. Thus a state of the electromagnetic field in quantum mechanics is described by a number of expandors, one for each Fourier component. By using the electromagnetic equation which gives the value of the divergence of the potentials, one can eliminate in a non-relativistic way the 0 component and one other component of each of the four-dimensional oscillators, so that only two-dimensional oscillators are left. This circumstance has made it possible for people to develop quantum electrodynamics without using expandors.

Another possible application of expandors is to the spins of particles. The wave function describing a particle may be a function of the four co-ordinates \( x_\mu \) of the particle in space-time and also of the four variables \( \xi_\mu \) whose coefficients are the components of an expensor. As a simple example of relativistic wave equations for such a particle in the absence of external forces, one may consider

\[
\left( \hbar^2 \frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2 \right)\psi = 0, \quad \xi_\mu \frac{\partial}{\partial \xi_\mu} \psi = -\psi, \quad \xi_\mu \frac{\partial}{\partial \xi_\mu} \psi = 0.
\]

The first of these is the usual equation for the motion of the particle as a whole. The second shows that at each point in space-time the wave function \( \psi \) is homogeneous in the \( \xi \)'s of degree \(-1\). The third shows that the state for which the momentum-energy four-vector of the particle has the value \( p_\mu \) is represented by the wave function

\[
\psi = \frac{m}{\xi_0 p_0 - (\xi p)} e^{-i(p_0 x_0 - (\xi x))/\hbar}
\]

in three-dimensional vector notation. This may be expanded as

\[
\psi = m e^{-i(p_0 x_0 - (\xi x))/\hbar} \sum_{n=0}^{\infty} \frac{(\xi p)^n}{\xi_0^{n+1} p_0^{n+1}}.
\]

For the state for which the particle is at rest, \( p_1 = p_2 = p_3 = 0, p_0 = m, \) and \( \psi \) reduces to

\[
\psi = e^{-i m x_0/\hbar} \xi_0^{-1}.
\]
This ψ is spherically symmetrical, showing that when the particle is at rest it has no spin. But when the particle is moving, it is represented by the general ψ (34) and has a finite probability of a non-zero spin. In fact, taking for simplicity $p_2 = p_3 = 0$, the particle has a probability $(mp^2/p_0^2 + 1)^2$ of being in a state of spin corresponding to the transformations of $\xi_\alpha^\mu$ under three-dimensional rotations.

This example shows there is a possibility of a particle having no spin when at rest but acquiring a spin when moving, a state of affairs which was not allowed by previous theory. It is desirable that the new spin possibilities opened up by the present theory should be investigated to see whether they could in some cases give an improved description of Nature. The present theory of expansors applies, of course, only to integral spins, but probably it will be possible to set up a corresponding theory of two-valued representations of the Lorentz group, which will apply to half odd integral spins.

**Appendix**

The rules (5), (11) for forming scalar products are not always convenient for direct use. There are various ways of transforming them and making them more suitable for practical application. One such way has been given (Dirac 1942, equation 322) for the case of a single $\xi$ with ascending power series. Another way, applicable to the case of homogeneous functions of $\xi_1, \xi_2, \xi_3$, is provided by the following.

By partial integration with respect to $\xi'$, one gets, for $m > 0$,

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^m \xi'^n e^{i\xi \xi'} d\xi d\xi' = \int_{-\infty}^{\infty} \xi^{m-1} d\xi \left[ -i \xi'^n e^{i\xi \xi'} \right]_{\xi = -\infty}^{\xi = \infty} + i \int_{-\infty}^{\infty} \xi^{n-1} e^{i\xi \xi'} d\xi.
$$

If the integrals are made precise in the sense of Cesaro, which means neglecting oscillating terms like $\xi'^n e^{i\xi \xi'}$ for $\xi'$ infinite, this gives

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^m \xi'^n e^{i\xi \xi'} d\xi d\xi' = i^n \int_{-\infty}^{\infty} \xi^{m-1} \xi'^n e^{i\xi \xi'} d\xi d\xi'.
$$

Taking $m \geq n$ and applying the partial integration process $n$ times, one gets

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^m \xi'^n e^{i\xi \xi'} d\xi d\xi' = i^n n! \int_{-\infty}^{\infty} \xi^{m-n} e^{i\xi \xi'} d\xi d\xi' = 2\pi i^n n! \int_{-\infty}^{\infty} \xi^{m-n} \delta(\xi) d\xi = 2\pi i^n n! \delta_{mn}.
$$

It follows that if $A$ and $B$ are homogeneous functions of $\xi_1, \xi_2, \xi_3$ of degree $u$, their scalar product according to (5) is

$$
(A B) = (2\pi)^{-3/2} i^{-u} \int_{\cdots} A(\xi_1 \xi_2 \xi_3) B(\xi_1' \xi_2' \xi_3') e^{i(\xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3')} d\xi_1 d\xi_2 d\xi_3 d\xi_1' d\xi_2' d\xi_3'.
$$
As an application of this rule, take

\[ A = (\xi_1^2 + \xi_2^2 + \xi_3^2) S_{-2r}, \quad B = (\xi_1^2 + \xi_2^2 + \xi_3^2) S_{-2s}, \]

where the \( S \)'s are solid harmonic functions. Then, using three-dimensional vector notation, (36) gives

\[ (AB) = (2\pi)^{-3} i^{-n} \int \cdots \int \xi^{2r} \xi^{2s} S_{-2r}(\xi) S_{-2s}(\xi') e^{i(\theta \xi')} d\xi_1 \cdots d\xi_3. \]  

(37)

From Green's theorem

\[ \int \int \int \left( e^{i(\theta \xi')} \left( \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} \right) \left( \xi^{2r} S_{-2r}(\xi) \right) - \xi^{2r} S_{-2r}(\xi) \left( \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} \right) e^{i(\theta \xi')} \right) d\xi_1 d\xi_2 d\xi_3 \]

equals a surface integral of an oscillating kind which is to be counted as vanishing at infinity. This result reduces to

\[ 4r(u - r + \frac{1}{2}) \int \int \int_{-\infty}^{\infty} \xi^{2(r-1)} S_{-2r}(\xi) e^{i(\theta \xi')} d\xi_1 d\xi_2 d\xi_3 + \xi \int \int \int_{-\infty}^{\infty} \xi^{2r} S_{-2r}(\xi) e^{i(\theta \xi')} d\xi_1 d\xi_2 d\xi_3 = 0. \]

(38)

so (37) becomes, for \( r, s > 0, \)

\[ (AB) = (2\pi)^{-3} i^{-u+2} 4r(u - r + \frac{1}{2}) \int \cdots \int \xi^{2(r-1)} \xi^{2(s-1)} S_{-2r}(\xi) S_{-2s}(\xi') e^{i(\theta \xi')} d\xi_1 \cdots d\xi_3. \]  

(39)

Now suppose \( s \geq r \) and apply the procedure by which (37) was changed to (38) \( r \) times. The result is

\[ (AB) = (2\pi)^{-3} i^{-u+2r} 4r! (u - r + \frac{1}{2})! (u - 2r + \frac{1}{2})!^{-1} \]

\[ \times \int \cdots \int \xi^{2(s-r)} S_{-2r}(\xi) S_{-2s}(\xi') e^{i(\theta \xi')} d\xi_1 \cdots d\xi_3. \]  

(39)

where \( n! \) means \( \Gamma(n+1) \) for \( n \) not an integer. If \( s > r \), the procedure can be applied once more, and then shows that

\[ (AB) = 0 \quad \text{for} \quad r \neq s. \]  

(40)

If \( s = r \), (39) shows that

\[ (AB) = c i^{2r} (u - r + \frac{1}{2})!, \]  

(41)

where \( c \) depends only on \( u - 2r \) and on the two \( S \) functions.

**Reference**