Three-particle symmetry classifications according to the method of Dirac

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It is shown that the method of Dirac is an effective way of teaching three-particle symmetry classifications to students in the first-year quantum mechanics class, who wish to understand the symmetry properties associated with elementary particles before or without learning group theory. According to the method spelled out for the N-particle case by Dirac in Secs. 55 and 56 of his classic book *Principles of Quantum Mechanics*, explicit calculations are carried out for three particles. It is shown that this method leads to the three-particle wave functions and their symmetric combinations as given by Feynman *et al*. In the following paper, the general principles derived here are applied to the quark model of hadrons.

I. INTRODUCTION

In a previous paper by two of us,[1] we discussed the possibility of introducing basic high-energy hadronic properties in the first-year quantum mechanics course, without using the techniques of quantum field theory. At that time, we emphasized that this is now possible because there exists a simple theoretical model for this purpose, and because it is not yet clear whether the field theoretical approach is suitable for understanding relativistic bound-state problems.

One of the basic hadronic features is the resonance mass spectrum. It is now widely believed that baryons are bound states of three quarks, and that mesons consist of a quark and an antiquark. At present, both the meson and baryon mass spectra are consistent with energy level calculations based on nonrelativistic bound-state quantum mechanics.[2] Because of its mathematical simplicity, the harmonic oscillator potential has been very useful for the calculation of these mass spectra.

In addition to the potential governing forces between the quarks, we have to know how to handle spins and unitary spins in order to understand the perturbing forces which remove the degeneracies in the energy spectrum caused by the potential. The key problem in handling this degenerate perturbation theory is to make proper linear combinations of wave functions which diagonalize the energy matrix. In the case of mesons, it is a two-body problem and can be handled by the methods available in standard textbooks on quantum mechanics. For the baryons, we have to understand the symmetry problem of three particles. This three-body symmetry is easy for those who have had a course in group theory, but it is not necessarily trivial for students taking the first-year quantum mechanics course.

Since the baryon consists of three quarks, we have to consider an exchange degeneracy, which is analogous to the case of many electrons. As Dirac pointed out clearly in Secs. 55 and 56 of his classic book on quantum mechanics,[3] the Hamiltonian should be invariant under the exchange of quarks, and *physical states should therefore be eigenstates of the permutation operators whose eigenvalues correspond to constants of motion*. It is by now a *firmly established experimental law* that the baryonic states are totally symmetric under the exchange of the constituent quarks. In the present quark model, the quarks carry spin, unitary spin, and spatial coordinate quantum numbers. The problem is how to combine wave functions corresponding to these quantum numbers to make the overall baryonic wave function totally symmetric. In this connection, we note that Feynman *et al.*[4] gave a comprehensive calculational guide to this problem. The purpose of the present paper is to show that the explicit calculation for the three-quark system given in Ref. 4 can indeed serve as an excellent illustrative example for Dirac's approach to many-particle symmetry problems.

In Sec. II, we carry out an explicit calculation for three particles following the procedure spelled out by Dirac for the general case. The permutation operators which commute with the Hamiltonian and with one another have been constructed and their eigenvalues are calculated. In Sec. III, we construct wave functions which are diagonal in the permutation operators defined by Dirac. It is shown that this method leads to the three-quark wave functions given by Feynman *et al.*[4] Section IV deals with the problem of combining two symmetrized wave functions. It is shown that the combination formula given by Feynman *et al.* is derivable from the treatment Dirac gives in his book.

In the following paper,[5] we use the results of Secs. III and IV to work out the hadronic multiplets in the quark model.

II. APPLICATION OF DIRAC'S TREATMENT OF PERMUTATIONS TO THREE-PARTICLE SYSTEMS

In Secs. 55 and 56 entitled "Permutations as Dynamical Variables" and "Permutations as Constants of Motion," respectively, Dirac clearly spelled out his original ideas about the dynamical roles permutations play in quantum mechanics. The purpose of the present section is to work out a concrete illustrative example which might be helpful in understanding Dirac's original treatment. We shall carry out explicit calculations for the three-particle system.

Let us consider three similar objects labeled as 1, 2, 3, respectively. For this system, we can perform six permu-

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tations. First, there are three permutations of the form

$$(12), \qquad (23), \qquad (31), \tag{1}$$

where each number is replaced by the succeeding number in the bracket, while the first one goes to the last position. In addition, there are two permutations of the form

$$(123), \qquad (132). \tag{2}$$

The above five permutations together with the identity form the six permutations which can be performed on the three objects.

As Dirac did in his Eq. (13) of Sec. 55, we construct the following operators for the three-body system:

$$X_1 = I,$$
$$X_2 = [(12) + (23) + (31)]/3, \tag{3}$$
$$X_3 = [(123) + (132)]/2,$$

where I is the identity operator. X_1 can also be written as

$$X_1 = [I + (12) + (23) + (31) + (123) + (132)]/6. \tag{4}$$

The above operators commute with every permutation, and therefore with one another.

If theHamiltonian is invariant under permutations, the above three X_i's can be simultaneously diagonalized. The next question is how to find eigenvalues for these operators. Here again, we follow the steps outlined by Dirac in his Eq. (14) of Sec. 56. By explicit calculation, we drive

$$X_1^2 = X_1, \qquad X_1X_2 = X_2, \qquad X_1X_3 = X_3,$$
$$X_2^2 = (X_1 + 2X_3)/3, \qquad X_2X_3 = X_2, \tag{5}$$
$$X_3^2 = (X_1 + X_3)/2.$$

Following Dirac's Eq. (15) of Sec. 56 for the general case, we consider the following *arbitrary* function of the X operators[6]:

$$B = X_1 + X_2 + X_3. \tag{6}$$

Then

$$B^2 = (11/6)X_1 + 4X_2 + (19/6)X_3, \tag{7}$$

$$B^3 = (10/4)X_1 + 13X_2 + (37/4)X_3. \tag{8}$$

Because X_1 is the identity operator, its eigenvalue is always 1. By eliminating X_2 and X_3 from Eqs. (6)–(8), we arrive at

$$B^3 - (9/2)B^2 + 5B - (3/2) = 0, \tag{9}$$

which is Dirac's Eq. (16) of Sec. 56 for the "arbitrary" B given in Eq. (6). The above cubic equation has three roots

$$B_1 = 3, \qquad B_2 = 1, \qquad B_3 = 1/2. \tag{10}$$

We can use each of these three numbers to calculate the left-hand side of Eqs. (6)–(8), which then become three simultaneous linear equations. The solutions to these linear equations will indeed be the eigenvalues for the X operators. They are given in Table I.

If the Hamiltonian is invariant under exchange of particles, one of the five nontrivial permutations can also be simultaneously diagonalized, because it commutes with X_1, X_2, and X_3. Let us choose this particular permutation to be

Table I. Eigenvalues of X_1, X_2, X_3, and P. There are three different sets of the eigenvalues, resulting in three different symmetry classifications. The choice of the B function in Eq. (6) was arbitrary. However, the eigenvalues of the operators $X_1, \ldots, P$ are independent of the form of the B function.

B	X_1	X_2	X_3	P	Symbol
B_1	1	1	1	1	S
B_2	1	−1	1	−1	A
B_3	1	0	−1/2	1	α
				−1	β

$$P = (23). \tag{11}$$

Other permutations which do not commute with the above P cannot be simultaneously diagonalized. Since $P^2 = I$, the eigenvalue of this operator has to be either +1 or −1. We are interested here in how these eigenvalues are distributed in Table I. For this purpose, we introduce the operator P' defined as

$$P' = 3X_2 - P = (12) + (31) \tag{12}$$

or

$$P + P' = 3X_2. \tag{13}$$

P and P' satisfy also the relation

$$P'P = 2X_3. \tag{14}$$

For the symmetry classification corresponding to B_1, the eigenvalues already found, together with Eqs. (13) and (14), allow only $P = 1$. For the B_2 case, $P = -1$. However, for the B_3 case,

$$P + P' = 0, \qquad P'P = -1. \tag{15}$$

P in this case can therefore have both values: +1 and −1. These results are given in Table I.

III. CONSTRUCTION OF SYMMETRIZED WAVE FUNCTIONS

Having achieved a general classification for the symmetry states of three particles, we can now give an example of this classification scheme by constructing a set of wave functions which demonstrate it. In this process, we shall derive the wave functions given by Feynman *et al.*[4] in their Appendix A.

Suppose we have three particles which can be in any of three quantum states x, y, and z, with one particle in state x, another in y, and another in z.[7] We can then write the general state for such a system as

$$\psi = a|xyz\rangle + b|yxz\rangle + c|xzy\rangle + d|zyx\rangle + f|zxy\rangle + g|yzx\rangle, \tag{16}$$

where $a, b, \ldots, g$ are the coefficients to be determined by the symmetry property of the wave function.

A state of classification S will obey

$$X_2\psi = X_3\psi = P\psi = \psi. \tag{17}$$

Explicit calculation shows that this requires

$$a = b = c = d = f = g. \tag{18}$$

This will lead to a totally symmetric wave function those explicit form is well known and is given in the paper of

Feynman *et al.* If we want an A state, we must have $-X_2\psi = X_3\psi = -P\psi = \psi$, which results in

$$a = -b = -c = -d = f = g. \quad (19)$$

This leads to a totally antisymmetric wave function whose form is also well known.

If we want an α state, we need $X_2\psi = 0$, $X_3\psi = -(1/2)\psi$, and $P\psi = \psi$. These conditions will lead to

$$a + f + g = 0, \quad a = c, \quad b = g, \quad d = f. \quad (20)$$

Hence, there are four equations and six unknowns. This means that there will be a two-dimensional subspace of the α state. We can pick two linearly independent α states as

$$|\alpha\rangle_1 = (1/2\sqrt{3})[|xyz\rangle + |xzy\rangle + |yxz\rangle + |yzx\rangle - 2|zxy\rangle - 2|zyx\rangle], \ |\alpha\rangle_2 = (1/2)[|xyz\rangle - |yzx\rangle + |xzy\rangle - |yxz\rangle]. \quad (21)$$

The first α state is the one given by Feynman *et al.*, and the second is orthogonal to it.

Lastly, for a β state, we want $X_2\psi = 0$, $X_3\psi = -(1/2)\psi$, and $P\psi = -\psi$. This results in

$$a + f + g = 0, \quad a = -c, \quad b = -g, \quad d = -f. \quad (22)$$

Here again, we have a two-dimensional subspace of possible states. We can pick the first β state to be that given by Feynman *et al.*, and one orthogonal to it to be the second β state:

$$|\beta\rangle_1 = (1/2)[|xyz\rangle - |xzy\rangle + |yxz\rangle - |yzx\rangle],$$
$$|\beta\rangle_2 = (-1/2\sqrt{3})[|xyz\rangle - |yxz\rangle - |xzy\rangle + |yzx\rangle + 2|zyx\rangle - 2|zxy\rangle]. \quad (23)$$

We now have a complete set of six linearly independent states which are contained in the four symmetry classifications S, A, α, and β.[7]

IV. SYMMETRIZED PRODUCTS OF SYMMETRIZED WAVE FUNCTIONS

As was stated in Sec. I, we have to combine spin, unitary spin, and spatial wave functions to construct the totally symmetric overall baryonic wave function.[5] For this purpose, we consider in this section products of two symmetrized three-particle states and derive the relations for the products given by Feynman *et al.*[4] in their Eq. (A2).

We are interested in a product of wave functions with values in two separate spaces, for example, spin space and unitary spin space. Our wave function will be of the form

$$|ab\rangle = |a\rangle|b\rangle, \quad (24)$$

where a and b represent quantum numbers in two separate spaces. The state $|ab\rangle$ can be made to conform to the symmetry classification scheme of Sec. III, since all the results of Secs. II and III were derived from the properties of the operators X_1, X_2, X_3, and P, without any assumptions about the form eigenstates would take. What we would like to do is to derive a complete set of states $|ab\rangle$ which conform to the symmetry classification from the symmetrized $|a\rangle$ and $|b\rangle$.

A permutation operator acting on $|ab\rangle$ will permute the particles in $|a\rangle$ and $|b\rangle$ in an identical manner. Thus we can write

$$X_1 = I = I_a I_b = X_{1a}X_{1b}, \quad (25)$$

$$X_2 = (1/3)[(12) + (23) + (31)] = (1/3)[(12)_a(12)_b + (23)_a(23)_b + (31)_a(31)_b], \quad (26)$$

$$X_3 = (1/2)[(123) + (132)] = (1/2)[(123)_a(123)_b + (132)_a(132)_b], \quad (27)$$

$$P = P_aP_b = (23)_a(23)_b. \quad (28)$$

We are interested here in expressing symmetrized wave functions $|ab\rangle$ in terms of symmetrized $|a\rangle$ and $|b\rangle$. The simplest way to attack this problem is to write the right-hand sides of Eqs. (25)–(28) in terms of the operators which are diagonal in the symmetrized a and b spaces and/or other simple operators. X_1 of Eq. (25) and P of Eq. (28) are already in the desired form. The remaining problem is to work out X_2 and X_3. For this purpose, we carry out first the following simple calculations:

$$X_{2a}X_{2b} = (1/3)X_2 + (2/3)X_2(X_{3b}) \quad (29)$$

or equivalently

$$X_{2a}X_{2b} = (1/3)X_2 + (2/3)X_2(X_{3a}), \quad (29')$$

and

$$X_{3a}X_{3b} = (1/2)X_3 + X_3(X_{3b}) - (1/2)X_{3a}, \quad (30)$$

or

$$X_{3a}X_{3b} = (1/2)X_3 + X_3(X_{3a}) - (1/2)X_{3b}. \quad (30')$$

Let us consider a state which is a product of an S state in the a space with an S state in the b space:

$$\psi = |a\rangle_S|b\rangle_S, \quad (31)$$

and look at what the relations given in Eqs. (29) and (30) tell us about the wave function ψ of Eq. (31). Clearly,

$$X_{ia}\psi = X_{ib}\psi = \psi \qquad i = 1, 2, 3, \quad (32)$$

so that

$$(X_{2a}X_{2b})\psi = (X_{3a}X_{3b})\psi = \psi, \quad (33)$$

giving

$$\psi = [(1/3)X_2 + (2/3)X_2(X_{3b})]\psi = X_2\psi \quad (34)$$

and

$$\psi = [(1/2)X_3 + X_3(X_{3a}) - (1/2)X_{3b}]\psi. \quad (35)$$

Hence

$$X_3\psi = \psi. \quad (36)$$

Also, from P of Eq. (28),

$$P\psi = \psi. \quad (37)$$

Thus we have established that $\psi = |a\rangle_S|b\rangle_S$ is an eigenstate of X_1, X_2, X_3, and P, and that the wave function of Eq. (31) is an S state: $|ab\rangle_S$.

In a similar manner, we can use Eqs. (29), (29′), (30), (30′), and (37) to show that the following states fall into the given symmetry classifications:

$$|a\rangle_S|b\rangle_S = |ab\rangle_S, \qquad |a\rangle_S|b\rangle_\alpha = |ab\rangle_\alpha,$$
$$|a\rangle_S|b\rangle_\beta = |ab\rangle_\beta, \qquad |a\rangle_S|b\rangle_A = |ab\rangle_A,$$
$$|a\rangle_A|b\rangle_S = |ab\rangle_A, \qquad |a\rangle_A|b\rangle_\alpha = |ab\rangle_\beta,$$
$$|a\rangle_A|b\rangle_\beta = |ab\rangle_\alpha, \qquad |a\rangle_A|b\rangle_A = |ab\rangle_S. \quad (38)$$

However, when ψ is taken as the product of an α or β state in the a space with an α or β state in the b space, the terms $X_2\psi$ and $X_3\psi$ appear in Eqs. (29), (29′), (30), and (30′) with zero coefficient. These equations thus reduce to the identities from which no information can be obtained. To handle these cases, we consider the operators which simply change an α to a β state, and vice versa. For this purpose, let us introduce the operator

$$R = (1/\sqrt{3})[(12) - (31)], \quad (39)$$

acting on $|a\rangle$, $|b\rangle$, or $|ab\rangle$ state. R has the following properties:

$$[X_i, R] = 0 \qquad i = 1, 2, 3; \quad (40)$$

$$PR = -RP; \quad (41)$$

and

$$R^2 = (2/3)[1 - X_3]. \quad (42)$$

From Eq. (41), we see that if a state $|a\rangle$ is an eigenstate of P_a, then $R_a|a\rangle$ will be an eigenstate of P_a with an eigenvalue opposite to that of $|a\rangle$. From Eq. (40), we see that $R_a|a\rangle$ will have the same eigenvalues under X_{1a}, X_{2a}, and X_{3a} as $|a\rangle$. The logic is the same for the b space.

Let us first consider the action of R upon an S state:

$$P(R|\,\rangle_S) = -R|\,\rangle_S \quad (43)$$

and

$$X_i(R|\,\rangle_S) = R|\,\rangle_S \qquad i = 1, 2, 3, \quad (44)$$

where $|\,\rangle$ can be $|a\rangle$, $|b\rangle$, or $|ab\rangle$. Thus $R|\,\rangle_S$ will be an eigenstate of X_1, X_2, X_3, and P with eigenvalues 1, 1, 1, and -1, respectively. From the argument of Sec. II, it is clear that no such state can possibly exist. We conclude, therefore, that $R|\,\rangle_S = 0$, and from a similar argument, $R|\,\rangle_A = 0$. This agrees with Eq. (42), which for S and A states reduces to $R^2 = 0$.

Next, we turn to the α and β states. R will again change the sign of the P eigenvalues, while leaving the X_1, X_2, and X_3 eigenvalues unchanged. This means that

$$P(R|\,\rangle_\alpha) = -R|\,\rangle_\alpha, \quad (45)$$

$$X_1(R|\,\rangle_\alpha) = R|\,\rangle_\alpha, \qquad X_2(R|\,\rangle_\alpha) = 0, \quad (46)$$

and

$$X_3(R|\,\rangle_\alpha) = -(1/2)R|\,\rangle_\alpha. \quad (47)$$

Thus $R|\,\rangle_\alpha$ will be a β state. Conversely, R operating on a β state will give an α state. Since $X_3 = -1/2$ for α and β states, we will have $R^2 = 1$, and

$$R^2|\,\rangle_\alpha = |\,\rangle_\alpha, \quad (48)$$

$$R^2|\,\rangle_\beta = |\,\rangle_\beta. \quad (49)$$

We can pick

$$|\,\rangle_\beta = R|\,\rangle_\alpha \qquad \text{and} \qquad R|\,\rangle_\beta = |\,\rangle_\alpha \quad (50)$$

uniquely, given $|\,\rangle_\alpha$.[8] In terms of the R operators acting on the a and b spaces, we can write X_2 of Eq. (26) as

$$X_2 = (1/2)P_aP_b + R_aR_b. \quad (51)$$

In order to derive a similar formula for X_3, we introduce the operator

$$R' = (1/3)^{1/2}[(123) - (132)]. \quad (52)$$

Here again

$$[R', X_i] = 0 \qquad i = 1, 2, 3, \quad (53)$$

$$R'P = -PR'. \quad (54)$$

As for R defined in Eq. (39),

$$RR' = -R'R = P - X_2. \quad (55)$$

Thus for any α state $|\,\rangle_\alpha$,

$$R(R'|\,\rangle_\alpha) = (P - X_2)|\,\rangle_\alpha. \quad (56)$$

Since $R^2 = 1$ from Eq. (42),

$$RR'|\,\rangle_\alpha = R^2|\,\rangle_\alpha. \quad (57)$$

Thus

$$R'|\,\rangle_\alpha = R|\,\rangle_\alpha. \quad (58)$$

From Eq. (55),

$$R'R|\,\rangle_\alpha = -|\,\rangle_\alpha. \quad (59)$$

Thus from Eqs. (58) and (59), we derive, for $|\,\rangle_\alpha$ and $|\,\rangle_\beta$ obeying Eq. (50),

$$R'|\,\rangle_\alpha = |\,\rangle_\beta \qquad \text{and} \qquad R'|\,\rangle_\beta = -|\,\rangle_\alpha. \quad (60)$$

In terms of the R operators acting on the a and b spaces, we can write X_3 of Eq. (27) as

$$X_3 = (1/4)[4X_{3a}X_{3b} + 3R'_aR'_b]. \quad (61)$$

Using the X_2 and X_3 operators given in Eqs. (51) and (61), respectively, together with X_1 and P of Eqs. (25) and (28), we can show that, with $|\,\rangle_\beta = R|\,\rangle_\alpha$,

$$\begin{aligned}|ab\rangle_S &= [|a\rangle_\alpha|b\rangle_\alpha + |a\rangle_\beta|b\rangle_\beta]/\sqrt{2},\\ |ab\rangle_\alpha &= [-|a\rangle_\alpha|b\rangle_\alpha + |a\rangle_\beta|b\rangle_\beta]/\sqrt{2},\\ |ab\rangle_\beta &= [|a\rangle_\alpha|b\rangle_\beta + |a\rangle_\alpha|b\rangle_\beta]/\sqrt{2},\\ |ab\rangle_A &= [-|a\rangle_\alpha|b\rangle_\beta + |a\rangle_\beta|b\rangle_\alpha]/\sqrt{2}.\end{aligned} \quad (62)$$

These are indeed the formulas given by Feynman *et al.*[4] in their Appendix.

As was pointed out in Ref. 4, the symmetrized wave functions given in Sec. III together with the combination formulas given in Eqs. (38) and (62) form the mathematical basis for constructing three-quark baryonic wave functions for spin, unitary spin, and harmonic oscillator excitations, and for constructing the total wave function by making symmetrized combinations.

V. CONCLUDING REMARKS

Since 1930, Dirac's book on quantum mechanics has been serving its unique role in teaching quantum mechanics. It is also well known that some of the sections of this original book are somewhat too difficult for students taking first-year quantum mechanics. The teacher's job is of course to remove or reduce this difficulty by working out concrete examples. The purpose of the present paper is to make a direct physical application of Dirac's method of constructing symmetrized many-particle wave function spelled out in Secs. 55 and 56 of his classic book. The physical application in this case is the quark model for baryons which are believed to be bound states of three quarks.

While carrying out this illustrative calculation, we noted that most of the formulas usable in the quark model are already given in the paper by Feynman *et al.*[4] While the authors of Ref. 4 did not explain how they derived their formulas, we point out in the present paper that they are derivable from Secs. 55 an 56 of Dirac's book.

As was demonstrated by Feynman *et al.*,[4] the formulas discussed here enable us to understand the quark model without a formal education in the representations of the SU(3) and SU(6) groups. In the following paper,[5] we shall use the mathematical results of Secs. III and IV to construct a theory of the quark model which can be included in the first-year quantum mechanics curriculum.

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Section II of this paper is based on the homework problem which one of us (YSK) did for the first-year quantum mechanics course in 1958 when he was a senior at Carnegie Institute of Technology (now called Carnegie–Mellon University). Michel Baranger was the instructor for this course. The material of Sec. II has been used several times by Y. S. Kim as an illustrative example for the permutation group in the first-year quantum mechanics class and in a group theory course for physics graduate students at the University of Maryland.

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[6] The ultimate purpose of this calculation is to obtain the eigenvalues of X_1, X_2, and X_3. These roots do not depend the choice of the form of B. For instance, another arbitrary form $B' = 7X_1 + 11X_2 + 17X_3$ would give the same eigenvalues for the X operators. For the simple case of three particles, we can obtain these eigenvalues without introducing the B function. However, we used this function in order to preserve the full flavor of Dirac's original treatment.

[7] For the three-electron system which was discussed by Schiff, the electron spin can be either up or down. Thus at least two of the quantum numbers x, y, and z have to be the same. For this case, the antisymmetric state A vanishes. The mixed symmetry states α_2 and β_2 either vanish or become linearly dependent on others. See L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968). For the three-particle system with three different quantum numbers, these second α and β states should be included in order that there be a complete set of six independent wave functions. The authors of Ref. 4 overlooked this point, and gave only four wave functions.

[8] The operator R establishes a one-to-one correspondence between the spaces of α and β states. The states of Eqs. (21) and (23) obey $R|\,\rangle_{\alpha_1} = |\,\rangle_{\beta_1}$, and $R|\,\rangle_{\alpha_2} = |\,\rangle_{\beta_2}$.