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## Entropy and Lorentz Transformations

Y. S. Kim

*Department of Physics, University of Maryland, College Park, Maryland  
20742*

E. P. Wigner

*Department of Physics, Princeton University, Princeton, New Jersey 08544*

### Abstract

As the entropy is a measure of our ignorance, the lack of our knowledge of time-like variables in relativistic quantum mechanics can be translated into an entropy. Within the framework of the covariant harmonic oscillator formalism for relativistic extended hadrons, the entropy can be calculated in terms of the hadronic velocity. The entropy in this case is not derived from thermodynamics or statistical mechanics, but purely from the incompleteness of information.



The entropy is a measure of our ignorance and is computed from the density matrix [1, 2]. The density matrix is needed when the experimental procedure does not analyze all relevant variables to the maximum extent consistent with quantum mechanics [3]. The purpose of the present note is to discuss a concrete example of the entropy arising from our ignorance in relativistic quantum mechanics.

Let us consider a bound state of two particles. For convenience, we shall call the bound state the hadron, and call its constituents quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed [4].

However, there are at present no quantum measurement theories to deal with the above-mentioned time-like separation. We shall study in the present paper how this ignorance is translated into the entropy. Within the framework of the covariant harmonic oscillator formalism [5, 6], it will be shown that the entropy increases as the hadron gains its speed. The entropy defined in this way is a more fundamental quantity than the hadronic temperature discussed recently by Han *et al* [7]. It is independent of the question of whether the temperature can be defined.

Let us consider a hadron consisting of two quarks. If the space-time position of two quarks are specified by  $x_a$  and  $x_b$  respectively, the system can be described by the variables

$$X = \frac{x_a + x_b}{2}, \quad x = \frac{x_a - x_b}{2\sqrt{2}}. \quad (1)$$

The four-vector  $X$  specifies where the hadron is located in space and time, while the variable  $x$  measures the space-time separation between the quarks. In the convention of Feynman *et al* [8], the internal motion of the quarks bound by a harmonic oscillator potential of unit strength can be described by the Lorentz-invariant equation

$$\frac{1}{2} \left( x_\mu^2 - \frac{\partial^2}{\partial x_\mu^2} \right) = \lambda \psi(x). \quad (2)$$

We use here the space-favored metric:  $x^\mu = (x, y, z, t)$ .

It is possible to construct a representation of the Poincaré group from the solutions of the above differential equation [6]. If the hadron is at rest, the

solution should take the form

$$\psi(x, y, z, t) = \psi(x, y, z) \left(\frac{1}{\pi}\right)^{1/4} \exp\left(\frac{-t^2}{2}\right), \quad (3)$$

where  $\psi(x, y, z)$  is the wave function for the three-dimensional oscillator with appropriate angular momentum quantum numbers. There are no excitations along the  $t$  direction. Indeed, the above wave function constitutes a representation of the  $O(3)$ -like little group for a massive particle [6, 9].

Since the three-dimensional oscillator differential equation is separable in both spherical and Cartesian coordinate systems,  $\psi(x, y, z)$  consists of Hermite polynomials of  $x$ ,  $y$ , and  $z$ . If the Lorentz boost is made along the  $z$  direction, the  $x$  and  $y$  coordinates are not affected, and can be dropped from the wave function. The wave function of interest can be written as

$$\psi^n(z, t) = \left(\frac{1}{\pi}\right)^{1/4} \exp(-t^2/2) \psi_n(z), \quad (4)$$

with

$$\psi^n(z, t) = \left(\frac{1}{\sqrt{\pi n! 2^n}}\right)^{1/2} H_n(z) \exp\left(\frac{-t^2}{2}\right),$$

where  $\psi^n(z)$  is for the  $n$ th excited oscillator state. The full wave function  $\psi(z, t)$  is

$$\psi_0^n(z, t) = \left[\frac{1}{\pi n! 2^n}\right]^{1/2} H_n(z) \exp\left\{-\left(\frac{z^2 + t^2}{2}\right)\right\}. \quad (5)$$

The subscript 0 means that the wave function is for the hadron at rest. The above expression is not Lorentz-invariant, and its localization undergoes a Lorentz squeeze as the hadron moves along the  $z$  direction [7].

It is convenient to use the light-cone variables to describe Lorentz boosts. The light-cone coordinate variables are

$$u = \frac{z + t}{\sqrt{2}}, \quad v = \frac{z - t}{\sqrt{2}}. \quad (6)$$

In terms of these variables, the Lorentz boost along the  $z$  direction,

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}, \quad (7)$$

takes the simple form

$$u' = e^\eta u, \quad v' = e^{-\eta} v, \quad (8)$$

where  $\eta$  is the boost parameter and is  $\tanh(v/c)$ . The wave function of Eq.(5) can be written as

$$\psi_0^n(z, t) = \left[ \frac{1}{\pi n! 2^2} \right]^{1/2} H_n \left( \frac{u+v}{\sqrt{2}} \right) \exp \left\{ - \left( \frac{u^2 + v^2}{2} \right) \right\}, \quad (9)$$

If the system is boosted, the wave function becomes

$$\psi_\eta^n(z, t) = \left[ \frac{1}{\pi n! 2^2} \right]^{1/2} H_n \left( \frac{e^{-\eta} u + e^\eta v}{\sqrt{2}} \right) \exp \left\{ - \left( \frac{e^{-2\eta} u^2 + e^{2\eta} v^2}{2} \right) \right\}, \quad (10)$$

As was discussed in the literature for several different purposes, this wave function can be expanded as [6]

$$\psi_\eta^n(z, t) = \left( \frac{1}{\cosh \eta} \right)^{(n+1)} \sum_k \left[ \frac{(n+k)!}{n!k!} \right]^{1/2} (\tanh \eta)^k \psi_{n+k}(z) \psi_n(t). \quad (11)$$

From this wave function, we can construct the pure-state density matrix

$$\rho_\eta^n(z, t; z', t') = \psi_\eta^n(z, t) \psi_\eta^n(z', t'), \quad (12)$$

which satisfies the condition  $\rho^2 = \rho$ :

$$\rho_\eta^n(z, t; z', t') = \int \rho_\eta^n(z, t; z'', t'') \rho_\eta^n(z'', t''; z', t') dz'' dt''. \quad (13)$$

However, there are at present no measurement theories which accommodate the time-separation variable  $t$ . Thus, we can take the trace of the  $\rho$  matrix with respect to the  $t$  variable. Then the resulting density matrix is

$$\begin{aligned} \rho_\eta^n(z, z') &= \int \psi_\eta^n(z, t) \psi_\eta^n(z', t) dt \\ &= \left( \frac{1}{\cosh \eta} \right)^{2(n+1)} \sum_k \frac{(n+k)!}{n!k!} (\tanh \eta)^{2k} \psi_{n+k}(z) \psi_{k+n}^*(z'). \end{aligned} \quad (14)$$

The trace of this density matrix is one, but the trace of  $\rho^2$  is less than one, as

$$\begin{aligned} Tr(\rho^2) &= \int \rho_\eta^n(z, z') \rho_\eta^n(z', z) dz dz' \\ &= \left( \frac{1}{\cosh \eta} \right)^{4(n+1)} \sum_k \left[ \frac{(n+k)!}{n!k!} \right]^2 (\tanh \eta)^{4k}. \end{aligned} \quad (15)$$

which is less than one. This is due to the fact that we do not know how to deal with the time-like separation in the present formulation of quantum mechanics. Our knowledge is less than complete.

The standard way to measure this ignorance is to calculate the entropy defined as [1, 2]

$$S = -Tr(\rho \ln(\rho)). \quad (16)$$

If we pretend to know the distribution along the time-like direction and use the pure-state density matrix given in Eq.(12), then the entropy is zero. However, if we do not know how to deal with the distribution along t, then we should use the density matrix of Eq.(14) to calculate the entropy, and the result is

$$\begin{aligned} S &= 2(n+1)[(\cosh \eta)^2 \ln(\cosh \eta) - (\sinh \eta) \ln(\sinh \eta)] \\ &\quad - \left( \frac{1}{\cosh \eta} \right)^{2(n+1)} \sum_k \frac{(n+k)!}{n!k!} \ln \left[ \frac{(n+k)!}{n!k!} \right] (\tanh \eta)^{2k}. \end{aligned} \quad (17)$$

In terms of the velocity v of the hadron,

$$\begin{aligned} S &= -(n+1) \left\{ \ln \left[ 1 - \left( \frac{v}{c} \right)^2 \right] + \frac{(v/c)^2 \ln(v/c)^2}{1 - (v/c)^2} \right\} \\ &\quad - \left[ 1 - \left( \frac{1}{v} \right)^2 \right] \sum_k \frac{(n+k)!}{n!k!} \ln \left[ \frac{(n+k)!}{n!k!} \right] \left( \frac{v}{c} \right)^{2k}. \end{aligned} \quad (18)$$

Let us go back to the wave function given in Eq.(5). As is illustrated in Figure 1, its localization property is dictated by the Gaussian factor which corresponds to the ground-state wave function. For this reason, we expect that much of the behavior of the density matrix or the entropy for the n-th

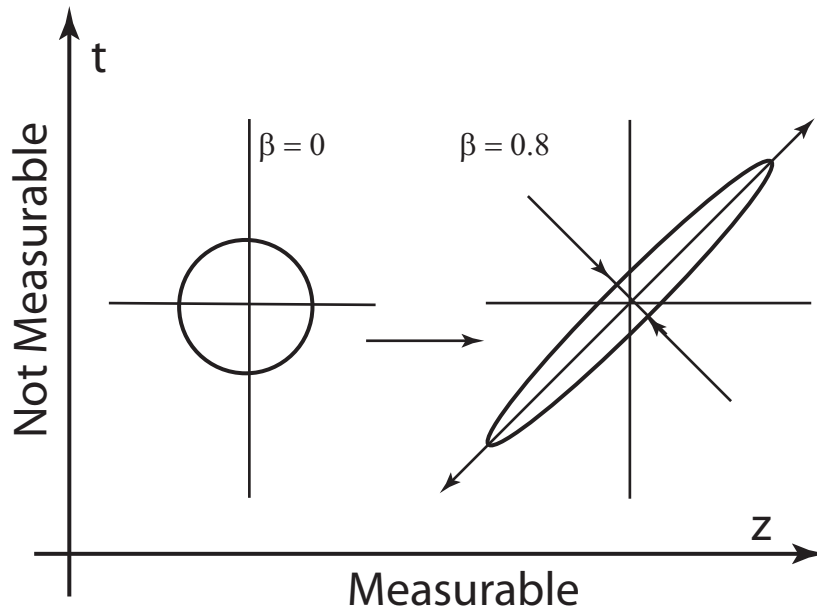


Figure 1: Localization property in the  $zt$  plane. When the hadron is at rest, the Gaussian form is concentrated within a circular region specified by  $(z + t)^2 + (z - t)^2 = 1$ . As the hadron gains speed, the region becomes deformed to  $e^{-2\eta}(z + t)^2 + e^{2\eta}(z - t)^2 = 1$ . Since it is not possible to make measurements along the  $t$  direction, we have to deal with the information less than complete.

excited state will be the same as that for the ground state with  $n = 0$ . For this state, the density matrix and the entropy are

$$\rho(z, z') = \left( \frac{1}{\pi \cosh(2\eta)} \right)^{1/2} \exp \left\{ -\frac{1}{4} \left[ \frac{(z + z')^2}{\cosh(2\eta)} + (z - z')^2 \cosh(2\eta) \right] \right\}, \quad (19)$$

$$S = 2 \left[ (\cosh \eta)^2 \ln(\cosh \eta) - (\sinh \eta)^2 \ln(\sinh \eta) \right], \quad (20)$$

respectively. The quark distribution  $\rho(z, z)$  becomes

$$\rho(z, z) = \left( \frac{1}{\pi \cosh(2\eta)} \right)^{1/2} \exp \left( \frac{-z^2}{\cosh(2\eta)} \right). \quad (21)$$

The width of the distribution becomes  $\sqrt{\cosh \eta}$ , and becomes wide-spread as the hadronic speed increases. Likewise, the momentum distribution becomes wide-spread [6, 7]. This simultaneous increase in the momentum and position distribution widths is called the parton phenomenon in high-energy physics [10]. The position-momentum uncertainty becomes  $\cosh$ . This increase in uncertainty is due to our ignorance about the physical but unmeasurable time-separation variable.

The use of an unmeasurable variable as a “shadow” coordinate is not new in physics and is of current interest [11]. Feynman’s book on statistical mechanics contains the following paragraph [12].

*When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

In the present paper, we have identified Feynman’s rest of the universe as the time-separation coordinate in a relativistic two-body problem. Our ignorance about this coordinate leads to a density matrix for a non-pure state, and consequently to an increase of entropy.

Finally, let us examine how the ignorance will lead to the concept of temperature. For the Lorentz-boosted ground state with  $n = 0$ , the density matrix of Eq.(21) becomes that of the harmonic oscillator in a thermal equilibrium state if  $(\tanh \eta)^2$  is identified as the Boltzmann factor [7]. For other states, it is very difficult, if not impossible, to describe them and thermal

equilibrium states. Unlike the case of temperature, the entropy is clearly defined for all values of  $n$ . Indeed, the entropy in this case is derivable directly from the hadronic speed.

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